

Jacques Sesiano

An Ancient Greek Treatise on Magic Squares

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I. General notions on magic squares

A *magic square* is a square divided into a square number of cells in which natural numbers, all different, are arranged in such a way that the same sum is found in each horizontal row, each vertical row, and each of the two main diagonals.

A square with n cells on each side, thus n^2 cells altogether, is said to have the *order* n . The constant sum to be found in each row is called the *magic sum* of this square.

Usually what is written in a square of order n , thus with n^2 cells, are the first n^2 natural numbers. Since the sum of all these numbers equals

$$\frac{n^2(n^2 + 1)}{2},$$

the sum in each row, thus the magic sum for such a square, will be

$$M_n = \frac{n(n^2 + 1)}{2}.$$

I. A square displaying this magic sum in the $2n + 2$ aforesaid rows is an *ordinary magic square* (Fig. 1). It meets the minimum number of required conditions, and such a square can be constructed for any given order $n \geq 3$ (a magic square of order 2 is not possible with different numbers).

1	32	34	3	35	6
30	8	27	28	11	7
19	23	15	16	14	24
18	17	21	22	20	13
12	26	10	9	29	25
31	5	4	33	2	36

Fig. 1

But there is an important, other kind.

II. A *bordered magic square* is one where removal of its successive borders leaves each time a magic square (Fig. 2). With an odd-order square, after removing each (odd-order) border in turn, we shall finally reach the smallest possible square, that of order 3. With an even-order square, that will be one of order 4 (its border cannot be removed since there is no

magic square of order 2). For order $n \geq 5$, bordered squares are always possible.

16	81	79	77	75	11	13	15	2
78	28	65	63	61	25	27	18	4
76	62	36	53	51	35	30	20	6
74	60	50	40	45	38	32	22	8
9	23	33	39	41	43	49	59	73
10	24	34	44	37	42	48	58	72
12	26	52	29	31	47	46	56	70
14	64	17	19	21	57	55	54	68
80	1	3	5	7	71	69	67	66

Fig. 2

As seen above, the magic sum for a square of order n filled with the n^2 first natural numbers is

$$M_n = \frac{n(n^2 + 1)}{2}.$$

Clearly, the *average* sum in each case is $\frac{n^2+1}{2}$. Accordingly, for m cells, the average sum should be m times that quantity; this will be called the *sum due* for m cells. Thus, the inner square of order m ($m \geq 5$) within a bordered square of order n must contain in each row its sum due, namely

$$M_n^{(m)} = \frac{m(n^2 + 1)}{2}$$

if the main square is filled with the n^2 first natural numbers. The sum in a row in one border will therefore differ from the next by $n^2 + 1$. From this it follows that pairs of elements of the same border which are horizontally and vertically (diagonally for the corner cells) opposite add up to $n^2 + 1$. Such pairs are what we call *complements*: to each ‘small’ number a (less than $\frac{n^2+1}{2}$) is associated the ‘large’ number $(n^2 + 1) - a$. For example, in each border of the above figure the sum of opposite elements is 82, while 41 belongs to the centre and $1, \dots, 40$ is the set of small numbers. We therefore infer that, for a bordered square of odd order filled with the first natural numbers, the central element must be $\frac{n^2+1}{2}$ (the *median*). The case of even-order squares is similar, except that there are two median numbers, $\frac{n^2}{2}$ and $\frac{n^2}{2} + 1$, to be placed in the central 4×4 square, the small numbers being those less than (or equal to) $\frac{n^2}{2}$.

A more particular kind of magic square is that of *composite* squares: the main square comprises subsquares which, taken individually, are also magic (Fig. 3). The possibility of such an arrangement depends on the divisibility of the order of the main square.

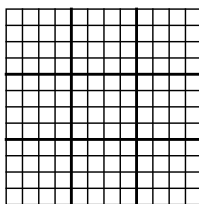


Fig. 3

There exist ways of directly constructing a square of given order and given type, that is, belonging to one of the above two kinds: once the empty square is drawn, a few, easily remembered instructions will enable us to place the sequence of consecutive numbers without any computation or recourse to trial and error. These are called *general construction methods*.

Now there exists no general method *uniformly applicable to any magic square*. Indeed, general methods are applicable to, at most, one of three categories of order, which are:

- The squares of *odd* orders, also called *odd squares*, thus with $n = 2k+1$, of which the smallest is the square of order 3.
- The squares of *evenly-even* orders, also called *evenly-even squares*, thus with $n = 4k$, of which the smallest is the square of order 4.
- The squares of *evenly-odd* orders, also called *evenly-odd squares*, thus with $n = 4k + 2$, of which the smallest is the square of order 6.

These methods are, by definition, applicable whatever the size of the order; but they may require some adapting for squares of lower orders, like 3 or 4, sometimes also 6 and 8.

Remark. Speaking of different *methods* suggests that a square of given order may take different aspects. As a matter of fact, the number of possible configurations (excluding mere inversions and rotations of the square) rapidly increases with the size of the order. Whereas there is just one form of the magic square of order 3, there are already 880 for the order 4, as discovered in 1693.¹

¹ By Frénicle de Bessy in his *Table generale des quarrrez de quatre*.

Note on the history of magic squares

The origin of the science of magic squares is unknown. It is commonly said, and generally accepted, that the earliest magic square appeared in China at the beginning of our era. Now, first, it is the square of order 3, and, second, higher-order squares involving general construction methods do not occur in China before the 12th century, and are clearly of Arabic or Persian origin. For there are from the tenth century onwards numerous texts in Arabic, later also in Persian, displaying a continuous development with earlier methods being extended or new ones invented.

Now trying to fix the starting point of this development led to a major problem: two tenth-century Arabic texts are preserved, one by a competent mathematician, Abū'l-Wafā' Būzjānī (940-997/8) and the other, with only a part on magic squares, by a less competent one, 'Alī ibn Aḥmad al-Anṭākī (d. 987). Their edition, though, shows that their state of knowledge was just in reverse proportion: whereas Būzjānī seemed to write the first steps in the construction of magic squares, and had only one general method, namely for constructing bordered odd-order squares, the second had general methods for all three types of bordered squares and applied them to quite elaborate cases. As the present author wrote in the edition of these texts, *the tenth century, for the (mediaeval) history of magic squares, gives the impression of being both a beginning and an end.*² The discovery of the text studied here, a translation of an anonymous Greek text, brought the solution: Anṭākī merely reproduced it, and, since it does not, for the case of bordered squares, give the foundation of these general methods, Būzjānī attempted to do so (succeeding only in the case of the odd orders).

As to the question of the first studies on magic squares, it is still unresolved, for those have yet to come to light — and probably never will. Our text does show that already in antiquity elaborate methods were being invented. But it leaves us in ignorance of the earlier history. Besides being anonymous, it does not refer to any person or treatise. We are only told about the existence of ways to construct ordinary magic squares, moreover not considered by our text (see its §2, or below, p. 165). In short, apart from this isolated treatise and this scarce information, we know nothing about the studies of magic squares in Greek antiquity.

Returning to Arabic times, we do see continuous development in the 11th and early 12th centuries, with the discovery of various constructions for ordinary magic squares. Būzjānī spares no effort in attempting to

² *Magic squares in the tenth century*, p. 8.

construct a few ordinary squares of small orders by individual methods, but without finding any general method. Indeed, general methods only begin to appear in the 11th century, first for odd and evenly-even orders; here the contribution of Ibn al-Haytham (ca. 965–1041) has proved to be essential. He considered the properties of the *natural squares*, that is, of the squares filled with the natural numbers taken in succession, and found that their main diagonals give the magic sum for the magic square of the same order while the sums displayed in opposite (horizontal and vertical) rows differ from the magic sum by equal amounts, but with opposite sign. Thus, rather than considering the squares by individual order, he sought (and found) ways to compensate the differences with exchanges between opposite rows. The less simple case of evenly-odd orders was finally solved towards the end of the 11th century.³

Squares filled with non-consecutive numbers also began to appear at that time. The origin of that has to do with the association of Arabic letters with numerical values — an adaptation of the Greek numerical system (Fig. 4); this adaptation appeared in early Islamic times, before the adoption of Indian numerals, but remained in use later. Thus to the letters of a word or the words of a sentence can be associated a set of numbers. So, assuming that the word or sentence is written in one row of a square (of which it thus determines the order), the task was then to complete the square numerically so that it would display in each row the sum in question — a mathematically interesting problem since this is not always possible.

ā	ḃ	ḡ	ḏ	ē	ḥ	z	ḥ	ṭ
۱	۲	۳	۴	۵	۶	۷	۸	۹
۱۰	۲۰	۳۰	۴۰	۵۰	۶۰	۷۰	۸۰	۹۰
۱۰۰	۲۰۰	۳۰۰	۴۰۰	۵۰۰	۶۰۰	۷۰۰	۸۰۰	۹۰۰
۱۰۰۰								

Fig. 4

³ About this development, in particular how general methods were discovered, and contributions of Ibn al-Haytham and his successors, see our *Magic squares, their history and construction from ancient times to AD 1600* (hereafter simply ‘*Magic squares*’), pp. 25–29, 51–56, 88–93; or earlier editions: *Les carrés magiques*, pp. 25–28, 49–51, 85–89; *Маг. квадраты*, pp. 33–37, 58–61, 96–100.

Whether original or inspired by earlier texts, quite an elaborate study on the construction of such squares with a set of given numbers not in arithmetical progression appeared already in the early 11th century. We do not know the author's name, but we do at least know his motivation: as he tells us in the introduction, he devoted himself to the study of magic squares in order to find relief from worries and preoccupations.⁴ This, apparently efficient, remedy ultimately resulted in a treatise in which, after teaching various methods for the construction of ordinary squares of odd and evenly-even orders and bordered squares of all three types, he constructs squares of orders 3 to 8 with a given sequence of numbers in a row, these numbers corresponding to sacred names or sentences. He makes no allusion to constructing amulets. But any reader might have thought about such a use.

General studies did not stop after the 11th century, although they tended to consist in finding other, simpler or more elegant methods of constructing squares. The treatises also became more numerous. This meant, however, that they less demanded of readers, the main concern being to teach methods without going into the mathematical background. This reached such a point that, as the 13th-century Persian author 'Abd al-Wahhāb Zanjānī tells us, he had to write, at the request of some friends, a treatise that was shorter than he would have wished, for they merely wanted to have practical rules for constructing magic squares of any order.⁵ And such indeed is the characteristic of later treatises: written in response to public demand, they were to teach methods aiming at results, without bothering the reader with questions of foundation or feasibility.

Meanwhile, however, popular use of magic squares as amulets grew steadily. Authors then followed this trend: many late, shorter texts are just not interested in teaching the construction of magic squares — to say nothing of their mathematical foundation. They give the figures of a few magic squares, most commonly squares of the orders 3 to 9 associated with the seven then known planets (including Moon and Sun) of which they embodied the respective, good or evil, qualities — abundantly described and commented in these texts. The reader is taught on what material and when he is to draw each of such squares; for both the nature of the material and the astrologically predetermined time of drawing are presumed to increase the square's efficacy. It merely remains to put this object in the vicinity of the chosen person, beneficiary or victim. This must have been of great use in solving personal or business problems.

⁴ See our *Un traité médiéval*, pp. 21 & (Arabic) 208.

⁵ See our *Herstellungsverfahren II, II'*.

Of such kind were the first Arabic texts translated into Latin in 14th-century Spain. A characteristic example of theory and application is the 5×5 square, attributed to Mars, which uses its (mostly unfavourable) properties.⁶

The figure of Mars when unfavourable means war and exactions. It is a square figure, five by five, with 65 on each side. If you wish to operate with it, take a copper plate in the day and hour of Mars when Mars is decreasing in size and brightness, or malefic and retrograding, or in any way unfavourable, and engrave the plate with this figure; and you will fumigate it with the excrement of mice or cats. If you place it in an unfinished building, it will never be completed. If you place it in the seat of a prelate, he will suffer daily harm and misfortune. If you place it in the shop of a merchant, it will be wholly destroyed. If you make this plate with the names of two merchants and bury it in the house of one of them, hatred and hostility will come between them. If you happen to fear the king or some powerful person, or enemies, or have to appear before a judge or a court of justice, engrave this figure as said above when Mars is favourable, in direct motion, increasing in size and brightness; fumigate it with one drachma ($= \frac{1}{8}$ ounce) of carnelian stone. If you put this plate in a piece of red silk and carry it with you, you will win in court and against your enemies in war, for they will flee at the sight of you, fear you and treat you with deference. If you place it upon the leg of a woman,⁷ she will suffer from a continuous blood flow. If you write it on parchment on the day and the hour of Mars and fumigate it with birthwort and place it in a hive, the bees will all fly away.

It was thus the arrival of such texts in late mediaeval Europe which first aroused interest in, and later led to the study of, such squares there. This incidentally explains the use of the term ‘magic’ — formerly also ‘planetary’, which we find still employed by Fermat.⁸ One of the two 4×4 squares thus transmitted is that used by Dürer in his *Melencolia* in order to date it (Fig. 5; 15 14 occupy the two median cells on the bottom). As to the mediaeval Arabic (perhaps originally Greek) denomination, ‘Harmonious arrangement of numbers’ (*wafq al-a‘dād*), which had a more mathematical connotation, it remained unknown, as well as the various constructions described in Arabic and Persian manuscripts.

⁶ This and other examples in our *Magic squares for daily life*. The magic square in question is that of our figure 19, p. 174 below.

⁷ Reverting to evil uses.

⁸ *Varia opera mathematica*, p. 176; or *Œuvres complètes*, II, p. 194.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Fig. 5

The earlier transmission towards the East was more fruitful. India and China received many more examples of squares, sometimes also construction methods. In the Byzantine Empire, a treatise was written at the very beginning of the 14th century by Manuel Moschopoulos. From a set of examples obviously received from some Persian or Arabic manuscript, he reconstructed a few methods for ordinary squares of odd and evenly-even orders. (He obviously does not know anything about the existence of magic squares in ancient Greece.)

European scholars of the fifteenth and sixteenth century did just the same with the squares received, so these initial studies met limited success. A notable exception is Michael Stifel (1487–1567), who gives instructions for constructing bordered squares of all orders;⁹ he does not mention any source, nor does he give himself any credit, and his methods are not exactly like the ones we know from Arabic sources.

European research had thus to start afresh. A return to studying their mathematical foundations was initiated by prominent mathematicians like Fermat and Euler, and this helped to dissociate these squares from their unfavourable reputation. Methods of construction were discovered, sometimes rediscovered. New categories were introduced later, such as the ‘bimagic’ or ‘trimagic’ square (one remaining magic when its numbers are replaced by their squares or cubes). Pandiagonal squares, with both main and broken diagonals magic, received greater attention than before.¹⁰ A recurrent problem was that of the number of given possibilities for a given order larger than 4 (above, p. 3, *Remark*), closely linked to that of classifying magic squares by category — which is still the subject of contemporary research.

⁹ See his *Arithmetica integra*, fol. 24^v–30^r; or *Magic squares*, pp. 149–150 & 154, 161–163 & 166–167, 170–171 & 173 (*Les carrés magiques*, pp. 125–126 & 130, 137–139 & 141–142, 146 & 148–149; *Маг. квадраты*, pp. 136 & 141, 148–150 & 153–154, 157–158 & 160–161).

¹⁰ One example below (p. 204). Other Arabic examples are given in our *Magic squares*, see *index* there.

II. The ancient work preserved

A. Manuscripts

Manuscript **A**

The University of Ankara, or, more precisely, its Faculty of Language, History and Geography (Dil ve Tarih-Coğrafya Fakültesi), received a donation from a well-read book collector, Ismail Saib Sencer (d. 1940), of more than ten thousand manuscripts divided into two collections. Among the first, one finds the manuscript I, 5311, of 84 leaves, written on paper in the 13th century (7th century of the hegira). It contains ten treatises, all on mathematical subjects (geometry, algebra, magic squares), and copied by the same hand. The first of these (fol. 1^r–36^r) belongs to a commentary on Nicomachos' *Introduction to arithmetic* by Anṭākī (above, p. 4). The text preserved, though, is not the whole of that commentary. For, as indicated by the title, that is only its *third book*, thus the third part of it, and also the last since the colophon clearly indicates that the work ends there.

The writing (*naskhī*) is quite legible, except in some worm-eaten places. There are between 18 and 24 lines on a page, mostly 20 or 21. The copyist sometimes used red ink, namely for some headings, numerical symbols and framing the figures; he did it later on, and that is why one correction is in red (*l.* 484 in this edition of the Arabic text). The copy is on the whole in good condition, and the copyist is responsible for only a few textual omissions which he partly or completely remedied upon rereading. We may, though, remark that there is some disorder in the Arabic text on the leaves 18 and 19, and that only half of fol. 18^r contains text. It appears that there has been an error in an earlier copy, just reproduced by the present copyist, together with, in red ink, the explanation (fol. 18^v, see p. 90, note to §39 of the Arabic text): *This part is left blank for the figure, where K is (written) in red on the left-hand page*; now the place referred to is not found in the present copy.

Nicomachos of Gerasa is a relatively late Greek mathematician (end of the 1st century). His *Introduction to arithmetic* (Ἀριθμητικὴ εἰσαγωγή) is extant in Greek and was widely known during late antiquity; some Greek commentaries on it have also survived. The *Introduction* had a notable influence later as well. It was first translated into Syriac, then into Arabic in the 9th century by Thābit ibn Qurra, and its modern edition has shown that the Arabic translation is quite faithful to the Greek original. The

rôle of the *Introduction* in mediaeval Europe is important as well. There it was, until the 12th century, the only surviving work of ancient Greek mathematics, through an adaptation by Boëtius (who, in the early 6th century partly summarized the original text and added remarks). In short, by means of Nicomachos' book, ancient Greek, Byzantine, Arabic and mediaeval European readers could learn some basic number theory. Indeed, according to its Greek meaning, *arithmetic* has to do with what we would call today number theory, and *not* arithmetical calculations.

This notoriety does not go, though, hand in hand with the mathematical level. It is in no way comparable to that of classical authors like Euclid (around -300) and Archimedes (around -250) — or also Diophantos (who, unlike the other two, lived after Nicomachos). With Nicomachos the science of mathematics seems to regress: an opinion becomes a theorem and a test serves as a demonstration. In short, Nicomachos' *Introduction*, though undoubtedly of use at the time to anyone wishing to acquire elementary notions of number theory, must have disappointed others expecting to be trained in mathematical thinking. It embodies perfectly what would be called today 'democratization of knowledge'.

Book III of Anṭākī's Commentary consists of three distinct parts without any connection between them.¹¹

The first part consists of an enumeration of definitions, statements of propositions and identities. The definitions and propositions are chiefly taken from Euclid's *Elements*, but only those relating to elementary number theory. There is also a set of arithmetical identities, some of which are taken from the algebraic books of al-Khwārizmī, c. 820, and Abū Kāmil, c. 880. This first part at times leaves us in doubt as to Anṭākī's competence. Certain formulations are strange or nonsensical, incomplete or imprecise; definitions of terms do not appear together with the first use of these terms or are omitted; a demonstration is wrong, as well as two assertions.

The second part is on the construction of magic squares. It is just copied from the translation of our ancient text, but without any mention of it, as if the author were Anṭākī himself.

The third subject is that of hidden numbers, that is, of guessing a number thought of by someone else. The latter is asked to mentally perform various computations with that number and communicate the result. The asker will then be able to work out what that number was.

¹¹ The reader interested in a detailed analysis is referred to its edition in *Magic squares in the tenth century*.

Comparing now the content of Nicomachos' work with that of Anṭākī's Commentary, we make the surprising discovery that there is no connection between the two works — at least not in Book III of said Commentary since that is all we know. For the first part, Euclid and not Nicomachos would rightly be indicated as the source for most propositions. The second part, on magic squares, which reproduces our text, has no connection whatsoever with Nicomachos.¹² Finally, although we guess from later sources that problems on hidden numbers existed in antiquity,¹³ they have nothing to do with Nicomachos' *Introduction*. In short, Anṭākī must have taken the essence of these subjects from other sources, as, by the way, his often weak mathematical knowledge obliges him to, and probably without much change, as his literal copying of the part on magic squares suggests.

Manuscript *Ḍ*

Manuscript *Ḍ* belongs to the Delhi collection preserved since 1876 in the India Office Library in London. The Delhi collection represents what remained in 1858 of the Mughal Imperial Library after parts had been given away, sold or seized during the 18th and 19th centuries. It consists of over 3500 volumes of manuscripts mainly in Arabic, Persian and Urdu. Of these, manuscript Delhi Arabic 110 contains two unrelated works, the first being the *Nukhbat al-fikr* by Ibn Ḥajar al-ʿAsqalānī (fol. 2^v – 27^r), a known expert on religious science (early 15th c.), and the other (fol. 28^r – 119^v), with the title *Kitāb dīwān al-ʿadad al-wafq*, a compilation on the science of magic squares going back to the second half of the 12th century. This last copy (now made available on line by the Qatar National Library) is a beautiful item, carefully copied and with the use of several colors both for titles and subtitles and to distinguish the parts of the magic squares represented. We do not know, however, the author of the original text, for both the beginning and end leaves are missing. It is divided into eight 'books' (*maqālāt*), covering the following subjects (fol. 28^r – 29^r contain part of the introduction and a — not very accurate — table of contents):

Book I (fol. 29^v – 35^v): Preliminaries

Book II (fol. 35^v – 58^v): Ordinary magic squares

Book III (fol. 58^v – 80^v): Bordered magic squares

Book IV (fol. 80^v – 93^v): Curiosities

Book V (fol. 93^v – 98^v): Composite squares with equal subsquares

¹² Although a 3×3 magic square is reported to have been in Nicomachos' *Introduction* (see below, p. 163), that is not the case.

¹³ *Magic squares in the tenth century*, pp. 15 and 185 *seqq.*

Book VI (fol. 99^r – 109^v): Composite squares with unequal parts

Book VII (fol. 109^v – 116^v): Magic cubes¹⁴

Book VIII (fol. 116^v – 119^v): Magic triangles.

The borrowings are indeed from various early Islamic mathematicians, such as Abū'l-Wafā' Būzjānī (940–997/8; above, p. 4), Ibn al-Haytham (ca. 965–1041; above, p. 5), Abū Ḥātim Muẓaffar Asfīzārī (d. before 1121/2), and his contemporaries 'Abdarrahmān al-Khāzinī and 'Umar Khayyām.¹⁵ Finally the author refers to a person variously named as al-Mufaḍḍal ibn Thābit al-Ḥarrānī (60^v; see below, p. 16*n*); al-Mufaḍḍal ibn Thābit ibn Qurra (28^{ar}, 80^v; p. 30*n*); al-Mufaḍḍal ibn Thābit (94^r, 95^v, 102^r, 107^v; pp. 124, 152*n*, 154*n*). From his name it appears that he was a Sabean, thus one of the group of persons educated in Greek culture and religion which produced numerous excellent translators of Greek works in the 9th century. He is doubtless related to the known translator Thābit ibn Qurra (836–901), perhaps his son.

In the edition of Anṭākī's text, the present writer, considering the weakness of some of Anṭākī's work and the high level of the part on magic squares, observed that this latter part suggested *a much earlier, possibly Greek time for the first discoveries*.¹⁶ This is fully confirmed by MS. *D*, which reproduces al-Mufaḍḍal ibn Thābit's introduction, omitted in Anṭākī's copy, which attests that we have here the translation of an ancient treatise. MS. *D* also reports examples of constructions of larger squares being alluded to in the translation but mostly omitted by Anṭākī. Thus, although lacking two main parts of our treatise (see below, pp. 13–14), MS. *D* does complete substantially the text transmitted by Anṭākī.

Since al-Mufaḍḍal ibn Thābit mentions in the introduction to his translation late ninth-century scholars as persons of former times, while his translation is copied by Anṭākī in the middle of the tenth century and commented during its second half, that translation must go back to the early tenth century.

Remark. Since a work on magic squares (*Risāla fi'l-'adad al-wafq*) is attributed by Ibn al-Qiftī and Ibn abī Uṣaybi'a (both 13th c.) to Thābit ibn Qurra,¹⁷ we first thought he might be the translator and so attributed it to him in our *Magic squares*. But it now seems evident

¹⁴ On this and the next particular subjects, see our *Magic squares*, pp. 271–273, 280.

¹⁵ On their works on magic squares, see *Magic squares*, index.

¹⁶ *Magic squares in the tenth centuries*, p. 9.

¹⁷ See A. Müller's edition of Ibn abī Uṣaybi'a's work, I, p. 220, or J. Lippert's edition of Ibn al-Qiftī's, p. 119.

that he has not written such a work (our translator was not aware of any Arabic writing on this subject, see his introduction) and that the work referred to by Ibn al-Qiftī and Ibn abī Uṣaybi'a must be our ancient text's translation.

B. Brief survey of the treatise

After the translator's introduction (§0), we find a few generalities on magic squares (§§1–2). Then follow the four parts of our treatise.

Part I (§§3–6).

The first part is on the construction of bordered squares of odd orders. The author explains, but without justification, how to place each number in each border considered successively. A 12th-century reader who had read several treatises on magic squares, and knew Anṭākī's treatise (but not its source), expresses his opinion as follows: *As for other (authors), among whom al-Anṭākī, I found that they (merely) said to put such and such a number in such and such a cell without explaining the reason for that, although this subject is difficult to understand and requires to be memorized.*¹⁸ This allusion to the tedious construction of usual bordered squares indeed confirms the impression left by our text. Furthermore, not only does it list all the places of smaller numbers but also those of their complements, which is superfluous since their places are *ipso facto* known. To avoid such verbosity a later author, Khāzinī (mentioned above, p. 12), represented a single odd-order square of seventh order with the places of the small numbers in its various borders (below, pp. 16*n* for the Arabic reference and 173 for the square). This being indeed sufficient, the author of the compilation preserved in MS. *Ḍ* decided to omit this part, presenting Khāzinī's table instead. He did the same for Part III, on the construction of even-order squares, for the same reason.

Part II (§§7–35).

This is by far the longest and most difficult part of the work. The text of it is preserved in both manuscripts (for *Ḍ*, in Book IV), while for the examples of larger squares *Ḍ* completes *Ḍ*. We are to construct bordered squares of odd orders displaying separation by parity, with the odd numbers being grouped together in a rhomb (actually an oblique square) within the main square.

This second part is doubtless one of the finest pieces of ancient mathematics. The construction presented is not only most subtle but is, unlike

¹⁸ See our *Une compilation arabe*, p. 163, lines 239–241 of the Arabic text.

the previous part, justified: every step of the intricate placing of the surrounding even numbers is explained, and we are never left in doubt about the foundation of the procedure. Obviously, this was a topic for advanced readers, beyond the reach of most, and the previous part was perhaps just a preparation for it since these constructions are required for placing the odd numbers.

Part III (§§ 36–43).

Here we are told how to fill bordered squares of both even orders (divisible by 4 or by 2 only). This can be explained more succinctly than for odd-order squares, and the explanations are clear. This part is, again, omitted in MS. *Ḍ* since *Khāzinī* gave instead a table, including the two types of even orders (see p. 212).

Part IV (§§ 44–54).

Just as Part I was preparatory to Part II, so Part III paves the way to the last section. We learn how to construct composite squares for even orders, divided either into identical or different even-order subsquares, or also rectangular parts with both dimensions even (and not meeting the diagonals); for evenly-odd orders, a central cross (thus meeting the diagonal) separates the subsquares. Here too, the placing is clearly explained and justified. This may have been the second main purpose of our treatise, also intended for advanced readers. As for the manuscripts, the situation is the same as for the second part: the text is reproduced in both (for *Ḍ*, in Books V–VI), but *Ḍ* omits most of the figures, in this case a sizable set.

III. Text and translation

Besides explanatory footnotes, the translation contains additional intermediate (incomplete) figures and tables to aid the reader. (A more complete analysis will be found in the General commentary, pp. 163–219 below). Round brackets mark our additions to the text, used mainly in the translation to clarify it, while square brackets, employed both in the Arabic text and the translation, enclose interpolations. As to the Arabic text, each section is located in the manuscripts (in addition, for $\mathcal{A}$, in the edition) and supplemented with a critical apparatus.

(0) كان أوّل معرفتي بأمر العدد الوفق جدول الثلاثة الذى ذكره
 نيقوماخس فى الارثماطيقى. ثمّ وقع الى ابى القاسم الحجازى جدول
 الاربعة الذى يبتدئ من الواحد بتفاضل واحد واحد الى ستة عشر وكان
 تعجّب منه. ثمّ وجدتُ جدول الستّة على ظهر كتاب اقليدس نسخه
 اسحاق (بن) حنين. ثمّ وقع الى كتاب فيه ثلاثة او اربعة جداول ممّا⁵
 دون العشرة. ثمّ وجدتُ فى خزانة من كتب شيوخنا كتابين قد أتت
 الارضة على اكثرهما حتّى لا يفهم منهما الا اليسير وكان المختصر منهما
 بخط الماهانى والورقة الاولى من الاكثر بخط الحسين بن موسى النوبختى.
 فنظرتُ فيهما فوجدتُ استخراجهما متعبًا جدًّا ولاح لى انه يتهيأ ان
 يستخرج بعض ما تلف من كلّ واحد منهما بما سلّم من صاحبه وبان¹⁰
 يقوم فى النفس المعنى فيقام اللفظ مقام اللفظة حتّى يصحّ العرض.

(1) العمل فى هذا النوع ان ترسم سطوحًا مربّعةً «عدّتها» من ضرب
 عدد فى مثله وتثبت فيها اعدادًا من الواحد الى جملة تلك المربّعات
 فيكون «ما فى» [جملة تلك] [اعنى اعداد] كلّ سطر منها موافقًا لما فى
 السطر الآخر.

(0) D: 60^v, 4-60^v, 11.

القسم الثانى من الوفق المحلّق فى اختصار ما اورده المفضّل بن ثابت الخزائى : tit. D (60^v, 1-4) على سبيل الحكاية دون البرهان. قال (...)

الارضه [الارضه 7 || حنين cod., add. in marg.: اسحق [اسحاق 5 || القسم [القاسم 2
 add. cod. فابتدأت ذلك واستعنتُ الله عليه [العرض 11 || cod.

وقال الخزائى كان فى تصنيفه طول اختصرناه مؤامرةً وجدولاً : (60^v, 11) Post haec scr.

(1) A: 9^v, 8-9^v, 10 (ed. lines 433-436).

النوع الثانى فى علم عدد الوفق وهو بابان. الباب الأوّل فى علم الاعداد الافراد : tit. A (9^v, 6-8)

13 || pr. scr. et del. عدد [مثله 13 || cod. شيت [ترسم 12 || cod. ان العمل [العمل 12
 cod. جمل [جملة]

(§0. Translator's introduction)

My first acquaintance with the subject of magic squares was the figure of (order) three mentioned by Nicomachos in the *Arithmetic*.¹⁹ Then Abū al-Qāsim al-Ḥijāzī came across a figure of four, (with the numbers) beginning with 1 and ending with 16 with constant difference 1, and was filled with admiration. Next I found a figure of six on the back of Euclid's book (of the *Elements*) in the handwriting of Ishāq ibn Ḥunayn. Afterwards I came across a book containing three or four figures, all less than (the order) ten. Then I found in the Library, among the books of the caliphs' collection, two books, for the greater part damaged by termites, so that one could understand just little of them; the summary note about them was in al-Māhānī's handwriting and the first page for the most part in Ḥusayn ibn Mūsā al-Nawbakhtī's handwriting. Then I examined them, found elucidating them very arduous, (but) it occurred to me that it might be possible to make sense of those parts which had been damaged in one by what had been preserved in the other and to restore the proper meaning by replacing a word by another until the account was correct.

(§1. Definition of a magic square)

The treatment for this kind consists in drawing square areas in a quantity (equal) to the product of an (integral) number by itself, in which one will put numbers from 1 to the quantity of these (small) squares so that <the content in> [the sum of these] [that is, the numbers] each row of them is concordant with the content in (any) other row.²⁰

(§2. Two ways of constructing magic squares)

Some people begin by placing these numbers according to the succession of the natural order, from 1 to the number of squares in the figure where they wish to construct the magic square.²¹ Then they move its

¹⁹ About the authors mentioned here, see Commentary, pp. 163–164. All except Nicomachos are 9th-century scholars, well known from Ibn al-Nadīm's *Fihrist*, a bio-bibliographical work written towards the end of the 10th century.

²⁰ In an earlier copy the lacuna of the text (in angular brackets) has been filled with two glosses (in square brackets), which specify that the sum of the numbers is meant; the second gloss clarifies the first. Note that it is not explicitly stated that the constant sum will also be found in the two main diagonals.

²¹ The *natural square* is a square of the same order as that of the square to be constructed, filled with the natural numbers taken in order (above, p. 5).

(2) فمن الناس مَنْ يبتدىء بوضع هذه الاعداد على توالى النظم الطبيعي من الواحد الى حيث بلغت <عدة مربعات> هذه الصورة التي يريد ان يرسم اعداد الوفق فيها ثمّ ينقل اعدادها ابداً بالزيادة في بعض السطور وبالنقصان في السطور المضادة لها ثمّ ينسق جميع السطور على أمر واحد. وهذا باب فيه صعوبة على المبتدىء.

20

ومن الناس مَنْ يعمله <على وجه> آخر اسهل من هذا.

(3) وهو امّا في صورة الافراد فبان يضع العدد الاوسط من تلك الاعداد [التي انتهى اليها الوضع] في <وسط> المربع. ويجعل على قطره عن يساره <من فوق> العدد الذي يليه اصغر منه وبازائه العدد الذي يليه اكبر منه على قطره. ثمّ يجعل العدد الاصغر الذي يلي ذلك العدد 25 الاصغر تحته ويجعل بازائه <العدد الاكبر الذي يلي ذلك العدد الاكبر> فوق العدد الذي يلي ذلك العدد الاوسط. ثمّ يجعل العدد الاصغر الذي يلي الثاني في بيت الزاوية اليمنى وبازائه في الزاوية العدد الاكبر <الذي يلي العدد الاكبر> الثاني. ثمّ يجعل فيما يلي هذا الاكبر العدد الاصغر الذي يلي الثالث اصغر منه وبازائه العدد الاكبر الذي يلي ذلك العدد 30 الاكبر الثالث. فيتمّ بذلك مربعة ثلثة في ثلثة وتكون على هذه الصورة.

ب	ط	د
ز	هـ	ج
و	ا	ح

(2) A: 9^v, 11-9^v, 16 (ed. ll. 437-442).

cod. سفق [ينسق 19 || cod. والنقصان] وبالنقصان 19 || cod. نظم الطبيعة [النظم الطبيعي 16

(3) A: 9^v, 16-10^r, 4 (ed. ll. 443-452).

cod. راويه (الراويه corr. ex) الايمن [الزاوية اليمنى 28 || cod. الوسط [الوسط 22

[ثلثة في ثلثة 31 || add. (v. supra) et del. [الباقي 29 || cod. راويه (post.) [الزاوية

cod. 3 في 3 || Quadratum: Pro numeris præb. cod. notas Indorum vel Arabum.

numbers — (which are) always in excess in some rows and in deficit in the rows opposite — and arrange all the rows in a certain manner. This is a method presenting difficulty for the beginner.

Other people proceed with that in a different, and easier, way.²²

(Bordered odd-order squares)

For the figure of odd (order, they do it) as follows.

(§3. Construction of the bordered square of order 3)²³

They put the median number of these numbers [which the placing has reached]²⁴ in the centre of the square. (Next) they place, diagonally to its left,²⁵ above, the number preceding it and, diagonally opposite to it, the number following it. Then they put the number preceding that smaller number underneath it, and opposite to it the number following that larger number, (namely) above the number which follows that median number. Then they place the number preceding the second (smaller number) in the (upper) right-hand corner cell and, opposite to it in the corner, the number following the second larger number. Next they place in the (cell) following (that of) this larger number the smaller number which precedes the third, and opposite to it the number following this third larger number. With this, the square of three by three is completed, and it will have the following aspect.

4	9	2
3	5	7
8	1	6

$$n = 3$$

²² By constructing successive borders, the description of which follows. ‘Some people’ (...) ‘other people’ seems to be τινὲς μὲν (...) τινὲς δὲ.

²³ Some words or expressions constantly used in constructing bordered squares must now be defined. If the numbers to be placed are $1, 2, \dots, n^2$, with n odd, the *median* number will be $\frac{n^2+1}{2}$, those preceding it are the *small(er)* numbers, the others the *large(r)* ones. To each ‘small’ number a corresponds a ‘large’ number $n^2 + 1 - a$, its *complement*. (Remember that in bordered squares, numbers complementing one another face each other in the same border, vertically, horizontally or, in the case of corner cells, diagonally.) In the subsequent description, the numbers are placed beginning with the median in the central cell, then — with the small numbers taken in descending sequence and thus their complements in ascending sequence — in the successive borders.

²⁴ This gloss alludes to the natural square (§2), although we have finished with it.

²⁵ Actually, here and throughout, left of the reader.

(4) فإما ان كان المربع خمسة في خمسة فانك تجعل في البيت الاوسط العدد الاوسط من خمسة وعشرين وهو ثلاثة عشر. ثم تجعل العدد التالي الثلاثة عشر اصغر منه وهو اثنا عشر في البيت الذي على قطره حيث جعلت الاربعة في مربع الثلاثة وتجعل بازائه [تلك الصورة] العدد الذي 35 يلي الثلاثة عشر اكبر منه على قطره. ثم تجعل العدد الذي <يلي> الاثنى عشر اصغر منه وهو احد عشر حيث جعلت الثلاثة [في تلك الصورة] وتجعل بازائه العدد الاكبر الذي يلي الاربعة عشر وهو خمسة عشر في موضع السبعة [في تلك الصورة]. وثبت العدد الذي يلي الاحد عشر اصغر منه في بيت الزاوية اليمنى من مربع ثلاثة الداخل وتجعل بازائه في 40 البيت الذي أثبت فيه الثمانية العدد الاكبر الذي يلي الخمسة عشر وهو ستة عشر. ثم تجعل في البيت الذي يلي الزاوية اليمنى [وهو عشرة] العدد الاكبر الذي يلي الستة عشر وهو سبعة عشر ثم بازائه العدد الذي يلي العشرة [وهو اصغر منها] وهو تسعة. فيتم بهذا العمل المربعة التي في وسط مربعة خمسة في خمسة ويبقى من المربعة ستة عشر بيتاً ومن 45 الاعداد ستة عشر عدداً.

فتثبت العدد الذي يلي التسعة دونها وهو ثمانية في البيت الاعلى الذي يلي بيت الزاوية اليسرى فوق الاثنى عشر والثمانية عشر في البيت الذي يلي الزاوية اليسرى في الصف الاسفل تحت الستة عشر. وثبت في 50 البيت الذي يلي بيت الزاوية اليسرى تحته الى جانب الاثنى عشر سبعة

(4) A: 10^r, 5-10^v, 11 (ed. ll. 453-477).

33-4 || ١٣ cod. || ثلاثة عشر 33 || cod. محله [تجعل] 32 || ٥ في ٥ [خمس في خمس] 32
 قطر [قطره] 34 || ١٢ cod. || اثنا عشر 34 || cod. البالى (الثانى pro) للثلاثة عشر [التالى الثلاثة عشر
 38 || ١١ cod. || احد عشر 37 || cod. اما عشر [الاثنى عشر 36-7 || cod. الى [اكبر] 36 || cod.
 ٧ [السبعة] 39 || ١٥ cod. || خمسة عشر 38 || ١٤ cod. || الاربعة عشر 38 || cod. بازاه [بازائه
 ستة 42 || ١٥ cod. || الخمسة عشر 41 || ٨ cod. || الثمانية 41 || ١١ cod. || الاحد عشر 39 || cod.
 ١٢ cod. || سبعة عشر 43 || ١٦ cod. || الستة عشر 43 || ١٠ cod. || عشرة 42 || ١٦ cod. || عشر
 ستة 45 || cod. المربع corr. ex [المربعة] 44 || ٩ cod. || تسعة 44 || cod. عشرة [العشرة] 44
 ٩ cod. || التسعة 47 || ١٦ cod. || ستة عشر 46 || add. supra [الاعداد] 46 || ١٦ cod. || عشر
 cod. || والمني عشر [والثمانية عشر 48 || (sic) cod. || الاثنى عشر 48 || ٨ cod. || ثمانية 47
 50 || (sic) cod. || الاثنى عشر 50 || ١٦ cod. || الستة عشر 49 || cod. الايسر [الاسفل] 49

(§4. Construction of the bordered square of order 5)

If the square is five by five, you put in the central cell the median number of 25, namely 13.²⁶ Then you place the number preceding 13, thus 12, in the cell (next) diagonally, where you had placed 4 in the square of three; opposite to it diagonally, you place [this figure] the number following 13.²⁷ Next you place the number preceding 12, thus 11, where you had placed 3 [in this figure], and you place opposite to it the number following 14, thus 15, in the place of 7 [in this figure]. You place the number preceding 11 in the (upper) right-hand corner cell of the inner square of three, and you place opposite to it, in the cell where 8 had been put, the number following 15, thus 16. Then you place in the cell following the (upper) right-hand corner [thus 10] the number following 16, thus 17, then opposite to it the number next to 10 [which precedes it]²⁸, thus 9. With this treatment, the square in the centre of the square of five by five is completed, and there remain, from the square, 16 (empty) cells and, from the numbers, 16 (available) numbers.

	12	17	10	
	11	13	15	
	16	9	14	

6	8	23	24	4
7	12	17	10	19
5	11	13	15	21
25	16	9	14	1
22	18	3	2	20

$$n = 5$$

You then put the number preceding 9, thus 8, in the upper cell next to the left-hand corner cell, above 12, and 18 in the cell next to the left-hand corner in the lower row, underneath 16. You put 7 in the cell next to the left-hand corner cell, underneath it, next to 12, and 19 in the opposite cell of the right-hand row, next to 10. You put 6 in the upper left-hand corner,

²⁶ Whereas the previous figure was inserted at the appropriate place in the text, the others (all but one: §7) appear together at the end of the main chapters. We shall put some of them where they belong, and also add some intermediate results (incomplete figures), as well as tables when, later on, series of numerical quantities are mentioned.

²⁷ The bracketed words had, presumably, once been written by a reader below the figure for $n = 3$ and came to be incorporated in the text here. His purpose was to specify to what the (also added) allusions ‘in this figure’ found twice below were referring.

²⁸ *following* (verb *talā*) is indeed used both for the ascending sequence and the descending one, but specified by indicating whether it is smaller or larger than the preceding one (we omit this in the translation since we use ‘preceding’ and ‘following’); there was no specification here, whence the reader’s addition.

وبازائه في الصّف الايمن تسعة عشر الى جانب العشرة. وتثبت ستّة في الزاوية العليا اليسرى فوق السبعة وبازائها في زاوية القطر عشرين. وتثبت خمسة في البيت الاوسط الايسر تحت السبعة وتثبت احدًا وعشرين في البيت الذي يوازيه تحت التسعة عشر من الجانب الايمن. وتثبت اربعة في بيت الزاوية العليا اليمنى فوق التسعة عشر وبازائه في قطره اثنين وعشرين. وتثبت ثلاثة في البيت الاوسط الاسفل تحت التسعة وبازائها في الصّف الاعلى ثلاثة وعشرين فوق السبعة عشر. وتثبت الاثنين في البيت الثانى من الصّف الاسفل تحت الاربعة عشر وبازائه في الصّف الاعلى اربعة وعشرين الى جانب الاربعة. وتثبت الواحد الباقي في البيت الثانى من الصّف الايمن فوق العشرين وبازائه في الايسر خمسة وعشرين. وتمت 60 مربّعة خمسة في خمسة.

(5) فان كانت اعداد المربّعة اكثر من خمسة في خمسة فاثبت العدد الاوسط في البيت الاوسط وما يتلوه حيث كان نظيره من مربّعة ثلاثة في ثلاثة وخمسة في خمسة في ذيك المربّعين. فاذا بقى الصّف الذى يحيط بهذه المربّعة والاعداد الباقية عمدت الى الاعداد الباقية القليلة (والكثيرة). 65 وأخذت العديدين الاولين منها > وهما زوجان ابداً اذا كان النسق من واحد بزيادة واحد واحد فتثبت القليل منهما في البيت الذى يلي الزاوية اليسرى العليا وبازائه فيما يلي الزاوية اليسرى السفلى العدد الزوج

52 || 6 ستّة 51 || 10 cod. || العشرة 51 || 19 cod. || تسعة عشر 51 || 7 cod. || سبعة
 21 cod. || احدًا وعشرين 53 || 7 cod. || السبعة 53 || 5 cod. || خمسة 53 || 20 cod. || عشرين
 اثنين 55-6 || 19 cod. || التسعة عشر 55 || 4 cod. || اربعة 54 || 19 cod. || التسعة عشر 54
 23 cod. || ثلاثة وعشرين 57 || 9 cod. || التسعة 56 || 3 cod. || ثلاثة 56 || 22 cod. || وعشرين
 59 || 24 cod. || اربعة وعشرين 59 || 14 cod. || الاربعة عشر 58 || 17 cod. || السبعة عشر 57
 25 || خمسة وعشرين 60 || 20 cod. || العشرين 60 || 1 cod. || الواحد 59 || 4 cod. || الاربعة
 cod. || 5 في خمسة في خمسة 61 || 5 cod.

(5) A: 10^v, 11-11^r, 6 (ed. ll. 478-492).

5 || خمسة في خمسة 64 || 3 في 3 || ثلاثة في ثلاثة 63-4 || 5 في 5 || خمسة في خمسة 62
 السفلى 68 || cod. على 66 || وهى 66 || المربع 64 || الصّف 64 || cod. في 5
 cod. اليسرى 76 || اليمنى 76 || قبل 76 || قبله 76 || quasi السفلى (sæpius) cod.

above 7, and 20 opposite to it, in the (other) corner (cell) of the diagonal. You put 5 in the middle left-hand cell, underneath 7, and you put 21 in the cell opposite to it, on the right-hand side, underneath 19. You put 4 in the cell of the right-hand upper corner, above 19, and opposite to it diagonally, 22. You put 3 in the lower middle cell, underneath 9, and opposite to it in the upper row 23, above 17. You put 2 in the second cell²⁹ of the lower row, underneath 14, and 24 opposite to it in the upper row, next to 4. You put the remaining (number) 1 in the second cell of the right-hand row, above 20, and 25 opposite to it on the left. The square of five by five is (now) completed.

(§5. Construction of bordered squares of larger odd orders)

8	12	10	45	46	48	6
11	18	20	35	36	16	39
9	19	24	29	22	31	41
7	17	23	25	27	33	43
47	37	28	21	26	13	3
49	34	30	15	14	32	1
44	38	40	5	4	2	42

$$n = 7$$

If the numbers of the square are more than five by five, (you proceed) thus. Put the median number in the central cell and the next ones where were put, in the squares of three by three and five by five, the corresponding ones. When there remain the border surrounding this (latter) square and the still unplaced numbers, you deal with the remaining small and large numbers. You take the first two of them³⁰ — which will always be even if the sequence (of numbers) proceeds from 1 by successive increments of 1 — and you put the small one (12) in the cell next to the upper left-hand corner and, in the opposite cell, next to the lower left-hand corner, the larger even number (38). Take next the two numbers following these two numbers, which will always be odd;³¹ put the small one (11)

²⁹ Counted from right to left, as in Arabic.

³⁰ On either side of the numbers already used, thus one ‘small’ and the corresponding ‘large’. For convenience, we shall add the numbers in round brackets.

³¹ This mention of parity serves no purpose other than, perhaps, to draw attention to the fact that the even numbers will mostly occupy the horizontal rows and the odd ones, the vertical rows.

الاکثر. ثمّ خذ العددين اللذين يليان هذين العددين وهما فردان ابدًا
 70 فاثبت القليل منهما في البيت الذي تحت الزاوية اليسرى في الصفّ
 الايسر والكثير بازائه في الصفّ الايمن. ثمّ خذ العددين اللذين يتلوان
 هذين وهما زوجان فاثبت القليل منهما في الجهة العليا الى جانب البيت
 الذي أُثبت فيه العدد <الزوج> الذي كان قبله والكثير بازائه في الجهة
 السفلى. ثمّ خذ العددين اللذين يليان هذين وهما فردان ابدًا فتثبت
 75 القليل منهما في الجهة اليسرى الى جانب البيت الذي أُثبت فيه العدد
 <الفرد> الذي كان قبله والكثير في الجهة اليمنى مقابلةً.

(6) ثمّ كذلك تثبت الاعداد الافراد <القليلة في الجهة اليسرى وما يقابلها
 من الكبرى> في الجهة اليمنى بازائها والازواج القليلة في <الجهة> العليا
 وبازائها في الجهة السفلى ما يقابلها من الكبرى حتى تنتهي في جميع
 80 الجهات الى البيت الاوسط. فاذا فعلت ذلك فخذ العددين اللذين يتلوان
 اخيرى الاعداد التي اثبتتها وهما زوجان ابدًا فاثبت القليل منهما في
 بيت الزاوية اليسرى العليا والكثير بازائه على قطره في البيت الايمن. ثمّ
 خذ العددين اللذين يليان هذين وهما فردان ابدًا فاثبت القليل منهما في
 البيت الاوسط الايسر والكثير بازائه في الايمن. ثمّ خذ العددين اللذين
 85 يتلوان هذين وهما زوجان فاثبت القليل منهما في <بيت الزاوية اليمنى
 العليا والكثير بازائه على قطره في البيت الايسر الاسفل. ثمّ خذ العددين
 اللذين يليان هذين وهما فردان ابدًا فاثبت القليل منهما في <البيت
 الاوسط الاسفل والكثير في البيت الاوسط الاعلى بازائه. ثمّ اعمد الى ما
 يبقى من الاعداد. [وبعد ذلك] فاجعل الاعداد الازواج القليلة في الصفّ
 90 الاسفل وما يقابل كلّ واحد بازاء صاحبه في الصفّ الاعلى واجعل
 الافراد القليلة في الصفّ الايمن وما يقابلها كلّ واحد بازاء صاحبه في

(6) A: 11^r, 6-11^v, 4 (ed. ll. 493-514).

[جميع 79 || البيت corr. ex الكبرى 79 || cod. جهه [الجهة 79 || cod. بارامها [بازائها 78
 82 || cod. (saepius) استها [اثبتتها 81 || cod. اخر [اخيرى 81 || add. et del. ذلك
 || cod. الايسر [الايمن 91 || cod. الايسر [الايمن 82 || cod. اليمنى الاعلى (sic) [اليسرى العليا

in the left-hand row, in the cell below the left-hand corner, and the large one (39) opposite to it in the right-hand row. Take next the two numbers following them, which will (always) be even; put the small one (10) on the top, next to the cell where the previous even number had been put, and the large one (40) opposite to it on the bottom. Next take the two numbers following them, which will always be odd; you put the small one (9) on the left, next to the cell where the previous odd number had been put, and the large one (41) opposite on the right.

(§6. More general description of the placing)

Next you put, in like manner, the small odd numbers on the left and the large ones which are their complements opposite to them on the right, and the small even (numbers) on the top and the large ones which are their complements opposite to them on the bottom, until you reach on all (four) sides the middle cell.³² Once you have done that, take the two numbers following the last two of the numbers placed, which will always be even; put the small one in the upper left-hand corner cell and the large one opposite to it diagonally, in the right-hand (corner) cell. Take then the two subsequent numbers, which will always be odd; put the small one in the left-hand middle cell and the large one opposite to it, on the right. Take then the two subsequent numbers, which will (always) be even; put the small one in the upper right-hand corner cell and the large one opposite to it diagonally, in the lower left-hand (corner) cell. Take then the two subsequent numbers, which will always be odd; put the small one in the lower middle cell and the large one opposite to it, in the upper middle cell. Deal next with the remaining numbers:³³ [and after that]³⁴ place then the small even numbers in the lower row and each of their complements in the upper row, opposite to its associate, and place the small odd (numbers) in the right-hand row and each of their complements in the left-hand row, opposite to its associate, until you have finished with all the numbers. Then you will see the (previously) empty square filled with numbers, all meeting this (required) condition.³⁵

By following the way of filling the squares of three and five and always placing the remaining (numbers) as for them, this method will lead you

³² Without filling it. We shall thus have reached 9 and 41 for order 7, 11 and 71 for order 9. The word we translate by ‘complements’, which in this case also means ‘facing them’ (*mā yuqābilhā*), might be ἀντικείμενοι (*ἀλλήλοισι*).

³³ Starting from the middle cells just filled.

³⁴ Some early reader wanted to emphasize the steps. See also §§9, 33, 34.

³⁵ See §1. It is implicit that, by the construction, the squares are bordered ones, thus that each inner square is itself a magic square.

الصفّ الايسر حتّى تأتى على جميع الاعداد. فيرتسم لك الاعداد حشو
المربع الاصفر ويصير كلّها على هذه الشريطة.

واذا سلكت طريق حشو مربعي ثلاثة وخمسة ورسمت الباقية بمثلها
دائمًا ابداً اديك تلك الطريقة الى حشو جميع المربعات الافراد كائنةً ما
كانت اذا حشوت الاوساط ثمّ فعلت في الاطراف كفعلك في مربع خمسة
مع مربع ثلاثة هذا اذا كان ترتيب الاعداد على توالى الاعداد بزيادة
واحد واحد.

[ورسمت 94 || 3 cod. وه [ثلاثة وخمسة 94 || cod. اذا [واذا 94 || cod. الايمن [الايسر 92
[ثلاثة 97 || cod. ه [خمسة 96 || cod. كائنا [كائنة 95 || cod. عليهما [بمثلها 94 || cod. وقسمت
3 cod.

و	مح	مو	مه	ى	يب	ح
لط	يو	لو	له	ك	يخ	يا
ما	لا	كب	كط	كد	يعط	ط
مج	لج	كز	كه	كج	يز	ز
ج	يخ	كو	كا	كح	لز	مز
ا	لب	يد	يه	ل	لد	مط
مب	ب	د	ه	م	لح	مد

د	كد	كج	ح	و
يط	ى	يز	يب	ز
كا	يه	يخ	يا	ه
ا	يد	ط	يو	كه
ك	ب	ج	بخ	كب

ب	ط	د
ز	ه	ج
و	ا	ح

(3 × 3) A : 23^r (ed. p. 309).

tit.: ثلاثة في ثلاثة منسوقة على الواحد في كلّ صفّ ١٥

Eandem fig. præb. D, fol. 61^r, cum tit.: لوح ثلاثة جميع ما فيه ٤٥ وكلّ سطر منه ١٥

(5 × 5) A : 23^r (ed. p. 309).

tit.: ه في ه منسوقة على الواحد في كلّ صفّ ٦٥

Eandem fig. præb. D, fol. 61^r, cum tit.: لوح خمسة كلّ صفّ منه ٦٥

(7 × 7) A : 23^r (ed. p. 309).

tit.: ٧ في ٧ منسوقة على الواحد في كلّ صفّ ١٧٥

Eandem fig. præb. D, fol. 61^r, cum tit.: لوح سبعة جميع ما فيه ١٢٢٥ وكلّ صفّ منه ١٧٥

to the filling of all odd squares, whatever their (order), provided you fill (first) the central (parts) and then deal with the outer parts as you did for the square of five relative to the square of three — as long as the number sequence is the natural one with successive increments of 1.³⁶

4	9	2
3	5	7
8	1	6

$$n = 3$$

6	8	23	24	4
7	12	17	10	19
5	11	13	15	21
25	16	9	14	1
22	18	3	2	20

$$n = 5$$

8	12	10	45	46	48	6
11	18	20	35	36	16	39
9	19	24	29	22	31	41
7	17	23	25	27	33	43
47	37	28	21	26	13	3
49	34	30	15	14	32	1
44	38	40	5	4	2	42

$$n = 7^{37}$$

³⁶ In fact, the placing of successive terms would be the same with numbers in any other arithmetical progression.

³⁷ In MS. *A* the first squares are filled with Indo-Arabic numerals. Since the squares are represented together, first the odd-order ones then the even-order ones, those containing Indo-Arabic numerals are that for $n = 3$, the two for $n = 5$ and $n = 7$ (here and pp. 76–77) and the first for $n = 9$, all the other keeping the original alphabetical numerals. Thus, obviously, a later change.

ح	ف	عح	عو	عه	يب	يد	يو	ى
سز	كب	سد	سب	سا	كو	كح	كد	يه
سط	نه	لب	نب	نا	لو	لد	كز	يخ
عا	نز	مز	لح	مه	م	له	كه	يا
عج	نط	مط	مح	ما	لط	لج	كج	ط
ه	يط	كط	مب	لز	مد	نخ	سج	عز
ج	يز	مح	ل	لا	مو	ن	سه	عط
ا	نخ	يخ	ك	كا	نو	ند	س	فا
عب	ب	د	و	ز	ع	سج	سو	عد

ى	قك	قيح	قيو	قيد	قيح	يد	يو	يخ	ك	يب
قح	كح	ق	صع	صو	صه	لب	لد	لو	ل	بط
قه	فز	مب	فد	فب	فا	مو	مح	مد	له	يز
قز	فط	عه	نب	عب	عا	نو	ند	مز	لج	يه
قط	صا	عز	سز	نخ	سه	س	نه	مه	لا	يخ
قيا	صع	عط	سط	سج	سا	نط	نخ	مج	كط	يا
ز	كه	لط	مط	سب	نز	سد	عج	قح	صز	قيه
ه	كج	لز	سج	ن	نا	سو	ع	فه	صط	قيز
ج	كا	عح	لح	م	ما	عو	عد	ف	قا	قيط
ا	صب	كب	كد	كو	كز	ص	قح	فو	صد	قكا
قى	ب	د	و	ح	ط	قح	قو	قد	قب	قيب

(9 × 9) $\mathcal{A}$: 23^v (ed. p. 310).

tit. : ٩ فى ٩; add. etiam: ٨٢ ($= n^2 + 1$) et ٣٦٩ ($= M_9$).

Eandem fig. præb. $\mathcal{D}$, fol. 61^r, cum tit. : ٣٦٩ وكلّ صفّ منه

(11 × 11) $\mathcal{A}$: 24^r (ed. p. 311).

tit. : ١١ فى ١١

Eandem fig. præb. $\mathcal{D}$, fol. 61^v, cum tit. : لوح احد عشر جملة بيوتہ ١٢١ وجميع ما فيه من الاعداد ٧٣٨١ وكلّ صفّ من صفوفه ٦٧١

(10, 7) صا || لح (7, 7) نخ || فز (11, 8) قز || حل (10, 9) فز || كج (10, 10) كح || صج (8, 10) صج
 (4, 2) فح || فا (2, 3) قا || (v. loculum supra) كط (2, 5) صز || قط (11, 7) قط || ما

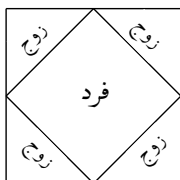
10	16	14	12	75	76	78	80	8
15	24	28	26	61	62	64	22	67
13	27	34	36	51	52	32	55	69
11	25	35	40	45	38	47	57	71
9	23	33	39	41	43	49	59	73
77	63	53	44	37	42	29	19	5
79	65	50	46	31	30	48	17	3
81	60	54	56	21	20	18	58	1
74	66	68	70	7	6	4	2	72

$$n = 9$$

12	20	18	16	14	113	114	116	118	120	10
19	30	36	34	32	95	96	98	100	28	103
17	35	44	48	46	81	82	84	42	87	105
15	33	47	54	56	71	72	52	75	89	107
13	31	45	55	60	65	58	67	77	91	109
11	29	43	53	59	61	63	69	79	93	111
115	97	83	73	64	57	62	49	39	25	7
117	99	85	70	66	51	50	68	37	23	5
119	101	80	74	76	41	40	38	78	21	3
121	94	86	88	90	27	26	24	22	92	1
112	102	104	106	108	9	8	6	4	2	110

$$n = 11$$

(7) وقد ترسم هذه المربعات رسماً آخر وهو ان تجعل الاعداد الافراد منها
 100 كلها في الوسط من المربعات يلي بعضها بعضاً والازواج في الاطراف.
 ورسم ذلك يكون بان ترسم مربعاً من ضرب عدد فرد في مثله كما
 فعلت فيما تقدّم ثمّ تعتمد الى الاعداد الفردية من الواحد الى نهاية
 الاعداد الفردية الكائنة في ذلك المربع فترتبها في وسط المربع حتى تجعلها
 كلها شبيهاً بالمعين في وسط «المربع» الكبير ويكون في المربع الكبير
 105 ابيات شبيهة بالمثلثات عدتها من كلّ جانب مساوية لعدتها من الجانب
 الآخر. فترسم فيها اعداداً ازواجاً من الاثنين الى آخر ما يكون في تلك
 المربعات وبنحو ان تكون الجمل متساويةً من كلّ الجهات. فتكون
 الاعداد الفردية حينئذٍ في داخل المربع الاعظم في صورة معيّنة والاعداد
 الزوجية مكتنفة بها وجميع جوانبه الاربعة على هذه الصورة.



(7) A: 11^v, 4-11^v, 15 (ed. ll. 515-526); D: 80^v, 22-81^r, 4.

المقالة الرابعة في الوفق الفرد الغريب المحلق (...) وهي تشتمل على : tit. D (80^v, 15-80^v, 22) :
 ثلاثة اقسام. القسم الاول في الفرد الغريب المحلق الذي ذكره المفضل بن ثابت بن قره في
 كتابه على سبيل الحكاية. قال

Post haec hab.: العدد الذي يكون له لبّ وهو في الاعداد الفردية فقط ومعناه ان نرسم لوح فرد
 ثمّ نعتمد من اوساط بيوت الصليب ونرسم مربعاً في اللوح الكبير زواياه في انصاف اضلاعه ويبقى
 في كلّ زاوية من زوايا المربع الكبير ابيات في شبيهه بالمثلث عدتها من كلّ جانب مساوية للجانب
 الآخر. فنرسم اعداد الزوج من الاثنين الى آخر الازواج في تلك الابيات الباقية وتكون الجمل
 متساوية من جميع الجهات فيكون الاعداد الفردية حينئذٍ في لبّ المربع والاعداد الزوجية مكتنفة
 لها من جميع الجهات الاربعة. وهذا ينقسم الى قسمين احدهما ما ترتيبه آ (!) هـ ط يـج والآخ
 ما ترتيبه جـ ز يـآ يـه وكل واحد منهما يختص بترتيب وضع سوى المربع الداخل منهما نرجع الى
 عمل واحد فيهما. ونعمل اولاً مربعات محاذية للمربع الكبير (infra, § 8) ثمّ نضيف الى كلّ واحد
 زواياها من اللبّ (§§ 9-11) ثمّ بعد الجميع نملاً اطرافها بالاعداد الازواج (§§ 12-35). والقول فيه
 يشتمل على ثلاثة ابواب.

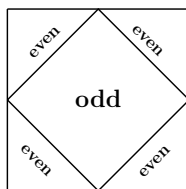
حسب [حينئذٍ 108 || A مساويه [متساويةً 107 || A مساوٍ [مساوية 105 || A مها [منها 99
 A. جوانبها [جوانبه 109 || A لها من جميع [بها وجميع 109 || A

(Bordered odd-order squares with separation by parity)

(§7. Introduction)

You may construct these (odd-order) squares in another way, namely by putting all odd numbers in the central part of the successive squares and the even ones in the outer parts.

The construction consists in this. You draw a square resulting from the multiplication of an odd number by itself, as you did before. Then you consider (first) the odd numbers from 1 to the last of those which will be in this square; you arrange them in the central part of the square so as to give them the shape of a rhomb within the main square.³⁸ This leaves in the main square (groups of empty) cells with the shape of triangles having on each (of the four) sides the same number of cells; then you write there (the) even numbers from 2 to the last to be (found) in these squares, and in such a way that the sums (in the main square) be equal everywhere. Thus doing, the odd numbers will be in the larger square within a rhombic figure and the even numbers will be enclosed by this (rhombic figure) and all four sides of the (square), as in this figure.³⁹



(Placing the odd numbers)

(§8. Filling the largest square within the rhomb)

The way to place them is as follows. You take 1 and the last term belonging to this square, namely its largest number, then 3 and the odd number preceding this largest one, and so on always until you reach their middle.⁴⁰

$$\begin{array}{ccccccc}
 1 & 3 & | & \dots & | & \frac{n^2-7}{2} & \frac{n^2-3}{2} & \frac{n^2+1}{2} \\
 n^2 & n^2-2 & | & \dots & | & \frac{n^2+9}{2} & \frac{n^2+5}{2} &
 \end{array}$$

³⁸ 'a rhomb'; in fact, an oblique square.

³⁹ This is the only other figure inserted in the text (see above, p. 21, *n.* 26).

⁴⁰ Grouping the odd numbers by pairs of complements, as in our table. The same will be done later for even numbers before placing them (§ 18).

- (8) فامّا كَيْفِيَّة اثباتها فهي ان تأخذ الواحد والطرف الاقصى من تلك
 110 المربّعة وهو اكثر عدد فيها ثمّ الثلاثة والعدد الذى يلى ذلك العدد الاكثر
 (من الافراد) ثمّ كذلك دائماً حتّى تنتهى الى الوسط منها.
 فتثبت العدد الاوسط فى البيت الاوسط من المربّعة وتثبت الطرفين
 اللذين يليانه من الافراد فى البيتين اللذين كنت قد اثبتت فيهما الطرفين
 اللذين يليان واسطة الاعداد التسعة التى فى مربّع ثلاثة فى ثلاثة ثمّ كذلك
 115 تفعل فى باقى الاعداد حتّى تأتى على مربّع ثلاثة فى ثلاثة ان كان عملك فى
 مربّع خمسة فى خمسة او فى مربّع سبعة فى سبعة. وان كان عملك للتسعة
 والاحد عشر فانك تفعل ذلك حتّى تأتى على مربّع خمسة فى خمسة الذى
 فى وسط كلّ واحد من هذين المربعين. وان كان للثلاثة عشر والخمسة عشر
 فانك تفعل ذلك حتّى تأتى على مربّع سبعة فى سبعة. وكذلك فى السبعة
 120 عشر والتسعة عشر ترسم الواسطة وما يتلوها من الاعداد الفردية فى مربّع
 تسعة فى تسعة. ودائماً تفعل ذلك حتّى تأتى على جميع المربّعة (التى فى
 الصورة) المعينة الداخلة.

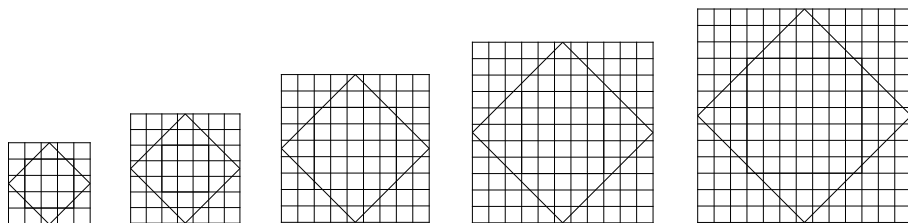
(8) A: 11^v, 16-12^r, 4 (ed. ll. 527-536); D: 81^r, 8-81^r, 15.

الباب الاول فى اثبات اعداد مربّعات فى اللبّ محاذية للوح الكبير: tit. D

اذا اردت ترتيب الاعداد فى المربّعات الفردية على هذه الشريطة [فامّا كَيْفِيَّة ... من المربّعة 110-3
 فخذ الواحد وقرينه والثلاثة وقرينها والخمسة وقرينها ثمّ كذلك حتّى تنتهى الى الواسطة فى مركز
 om. [تنتهى 112 || A] اكبر [اكثّر 111 || A] فهو [فهى 110 || D] ابيات المربّع
 114 || D [الواسطة sc.] يليانها [يليانه 114 || D] واثبت [وتثبت 113 || A] فيها [منها 112 || A]
 A فيها [فيها 114 || D] وثبت [قد اثبتت 114 || A] الدين [post.] اللذين 114 || D [الفرد [الافراد
 [ثلاثة فى ثلاثة 115 || D] التسعة من A, العدد فى التسعة الى هى [الاعداد التسعة التى فى 115 ||
 [عملك 116 || A] 3 فى 3 [ثلاثة فى ثلاثة 116 || D] om. [تفعل فى باقى الاعداد 116 || A] 3 فى 3
 [فى مربّع خمسة فى خمسة او فى مربّع سبعة فى سبعة 116-7 || A] add. فيها وكذلك ان كان عملك
 وان كان عملك ... تسعة 117-22 || D [للخمسة والسبعة A, فى مربّع 5 فى 5 او فى مربّع 7 فى 7
 على (add. in marg. هذه): add. D: [تسعة فى تسعة 122 || A] او فى مربّع 9 فى 9 [فى تسعة
 المقالة الثالثة فى: v. enim D, fol. 58^v; مراتبها كما ذكره فى المحلّق الفرد فى المقالة التى قبل هذه
 om. D. [ودائماً ... الداخلة 122-3 || الوفق المحلّق]

You put the median term in the central cell of the square. You put its two adjacent odd terms in the two cells where you had put, of the nine terms which were in the square of three by three, the two terms adjacent to the median.⁴¹ You do the same for the remaining numbers until you complete the square of three by three — if you are dealing with the square of five by five or the square of seven by seven.⁴² If you are dealing with (the squares of) nine and eleven, you do the same until you complete the square of five by five in the centre of these two squares. For the (squares of) thirteen and fifteen, you do the same until you complete the square of seven by seven. Likewise, for the (squares of) seventeen and nineteen, you place the middle (term) and its adjacent odd terms in the square of nine by nine. You will always do this until you complete the whole square contained by the inner rhombic figure.

$n = 2k + 1$	3	5	7	9	11	13	15	17	19
	1	3	3	5	5	7	7	9	9



(§9. Filling the remainder of the rhomb for the first three orders)⁴³

$n = 2k + 1$	5	7	9	11	13	15	17	19
	4 · 1	4 · 4	4 · 4	4 · 9	4 · 9	4 · 16	4 · 16	4 · 25

Once you have done that, take the two odd terms reached,⁴⁴ and put the small one on the left, in the middle cell of the border following the inner square you have (just) filled, and put the large one opposite to it on the right. Then put the following small number on the bottom, in the middle cell of the border following the inner square, and opposite to it

⁴¹ In all subsequent inner squares of order 3 the arrangement is, relative to that seen above (p. 19), inverted around the descending diagonal.

⁴² Our table gives the order of the largest square comprised by the rhomb if the main square is of order n . The subsequent illustrations show why successive pairs of orders each comprise a same largest central square.

⁴³ Thus $n = 5, 7, 9$. Placing of the remaining small odd numbers is common to the last two orders since the number of cells to be filled is the same. Likewise for the subsequent pairs of orders, as the table below shows.

⁴⁴ Reached without being placed. Thus 3 and 23 for the order 5.

(9) فاذا فعلت ذلك فخذ الطرفين اللذين تنتهى اليهما من الاعداد الفردية فاثبت القليل منهما فى الجهة اليسرى فى البيت الاوسط من الصف الذى يلي المربع الداخلى الذى حشوته واثبت الكثير بازائه فى الجهة اليمنى. ثم اثبت القليل الذى يتلوه فى الجهة السفلى فى البيت الاوسط من الصف الذى يلي المربع الداخلى وبازائه فى الجهة العليا الطرف الذى يقابله. فاذا فعلت ذلك فى خمسة فى خمسة فقد أتيت على جميع الافراد. فاذا كان عملك فى مربعى سبعة وتسعة فاحش المربع الداخلى واثبت الاعداد التى تنتهى اليها من الافراد كما عملت فى مربع خمسة حتى تنتهى الى هذا الموضع. ثم اثبت العدد القليل الذى تنتهى اليه [بعد ذلك] فى الجهة السفلى الى جانب البيت الاوسط يسراً وبازائه فى الجهة العليا الطرف الذى يقابله. ثم اثبت الطرف القليل الذى يتلو ذلك الطرف فى الجهة اليسرى الى جانب البيت الاوسط من فوق وبازائه فى الجهة اليمنى ما يقابله. ثم اثبت العدد القليل الذى يتلو ذلك العدد فى البيت الاوسط من الصف الثانى من الجهة اليسرى وبازائه فى الصف الثانى من الجهة اليمنى ما يقابله. ثم اثبت العدد القليل الذى يتلو ذلك العدد فى الجهة السفلى فى البيت الاوسط من الصف الثانى وبازائه فى الصف الثانى من الجهة العليا ما يقابله. ثم اثبت الطرف القليل الذى يتلو ذلك العدد فى الجهة العليا الى جانب البيت الاوسط من الصف الاول سمنه وبازائه <فى الصف الاول> من اسفل ما يقابله. ثم اثبت العدد القليل الذى يتلو ذلك الطرف فى الجهة اليمنى الى جانب البيت الاوسط من الصف الاول

(9) A: 12^r, 4-12^r, 16 (ed. ll. 537-555); D: 81^r, 18-81^v, 8.

tit. D : فصل. الاول: (l. 130) tit. hab. ante الباب الثانى فى حشو زوايا اللب :

[الكثير 126 || A (homoeotel.) om. فى الجهة اليسرى ... بعد ذلك 125-33 || A كذلك 124] 131 || D فان فاذا 130 || D pr. scr. et del. الذى اليمنى 127 || D hic et saepissime الكبير 134 || D العدد (post.) الطرف 134 || D فتره 133 || D حيث اثبتها من كما عملت فى يتلو 138 || A om. ثم اثبت ... ما يقابله 138-40 ('alif otiosum') AD saepius تلوا [يتلو سمنه وبازائه ... 141-43 || D العدد الطرف 140 || A om. اثبت 140 || D تلوه ذلك العدد من 144 || D om. الطرف 143 || D است اثبت 142 || A (homoeotel.) om. الصف الاول

on the top its complement. Once you have done that for the five by five (square), you will have completed (the placing of) all odd numbers.

		25		
	11	9	19	
3	21	13	5	23
	7	17	15	
		1		

			45			
		39	37	3		
	9	23	21	31	41	
7	15	33	25	17	35	43
	49	19	29	27	1	
		11	13	47		
			5			

				77				
			71	69	3			
		27	31	61	63	23		
	9	29	39	37	47	53	73	
7	15	25	49	41	33	57	67	75
	81	65	35	45	43	17	1	
		59	51	21	19	55		
			11	13	79			
				5				

If you are dealing with the two squares of seven and nine, fill the inner square, and put the odd numbers you have reached as you did for the square of five, until the same situation is attained.⁴⁵ Put then the small number you have reached [after that] (11) on the bottom, to the left of the middle cell, and, opposite to it on the top, its complementary term. Put then the following small term (9) on the left, just above the middle cell, and opposite to it on the right its complement. Then put the following small number (7) in the middle cell of the second border on the left and, opposite to it in the second border on the right, its complement. Then put the following small number (5) on the bottom, in the middle cell of the second border, and opposite to it, on the top of the second border, its complement. Then put the following small term (3) in the first border on the top, next to the middle cell on the right, and opposite to it, in the first border on the bottom, its complement. Then put the following small number (1) in the first border on the right, next to the middle cell, below it, and opposite to it on the left its complement. Once you have done that for these two squares, you will have reached 1 and the last (odd) term and you will have performed this (placing) in the desired way.

⁴⁵ Thus with placing the next two pairs in the middle cells of the next border, namely 15, 35 & 13, 37 (order 7), or 15, 67 & 13, 69 (order 9).

من اسفل وبازائه يسرةً ما يقابله. فاذا فعلت ذلك في هذين المربعين كنت
 145 قد انتهيت الى الواحد والطرف الاخير وأتيت ذلك على ما تريد.

(10) وان كان العمل في مربعي احد عشر وثلاثة عشر فاعمل مثل ذلك
 حتى تنتهي الى هذا الموضع. ثمّ ترسم العدد القليل الذي يتلو اخير
 الاعداد القليلة التي اثبتتها في الصف الاول من الجهة السفلى في البيت
 الثالث من الاوسط يسرةً وبازائه ما يقابله. ثمّ تثبت العدد القليل الذي
 150 يتلو ذلك في الصف الاول من الجهة اليسرى في البيت الثالث من الاوسط
 من فوق وبازائه في اليمنى ما يقابله. ثمّ تثبت الطرف القليل الذي يتلو
 ذلك في الصف الثاني من الجهة السفلى الى جانب البيت الاوسط والذي
 يقابله بازائه من الجهة العليا. ثمّ تثبت الطرف القليل الذي يتلو ذلك في
 الجهة اليسرى في الصف الثاني الى جانب البيت الاوسط والذي يقابله
 155 بازائه من الجهة اليمنى. ثمّ تثبت الطرف القليل الذي تنتهي اليه في
 البيت الاوسط من الصف الثالث الايسر وبازائه ما يقابله. ثمّ تثبت
 الطرف القليل الذي يتلو ذلك في البيت الاوسط من الصف الثالث اسفل
 وبازائه في الجهة العليا ما يقابله. ثمّ تثبت الطرف القليل الذي يتلو ذلك
 في الصف الثاني من الجهة العليا الى جانب البيت الاوسط وبازائه من
 160 الجهة السفلى ما يقابله. ثمّ تثبت العدد القليل الذي يتلو ذلك في الصف
 الثاني من الجهة اليمنى الى جانب البيت الاوسط وبازائه من الجهة اليسرى

فان كنت [ما تريد 145 || A فاست] وأتيت 145 || D الاخر [الاخير 145 || A من الاسفل] اسفل
 add. A. في مربع 9 فقد انتهت عملك

(10) A: 12^r, 16-12^v, 17 (ed. ll. 556-576); D: 81^v, 9-81^v, 18.

tit. D : فصل. الثاني :

فانك [فاعمل 146 || A وان كنت في مربع (sic) ١١ و ١٣] وان كان العمل ... وثلاثة عشر 146
 الذي A التي استها [التي اثبتتها 148 || AD اخر [اخير 147 || D المربع] الموضع 147 || D تفعل
 D العدد [الطرف 151 || D om. يسرةً ... من الوسط 149-50 || D الثاني] الثالث 149 || D ابها
 om. [الجهة اليسرى ... الذي يتلو ذلك 154-60 || D العدد [الطرف 153 || D جهة] الجهة 152
 161 || A add. in marg. A [اليه 155 || A pr. scr. et corr. الذي] والذي 154 || D (homoeotel.)
 164 || D om. [الجهة 163 || A om. العدد 162 || D om. [الجهة 161 || D الوسط] الاوسط

(§ 10. Filling the remainder of the rhomb for the orders 11 and 13)

					113					
				107	97	7				
			103	91	89	23	3			
		17	47	51	81	83	43	105		
	13	29	49	59	57	67	73	93	109	
11	27	35	45	69	61	53	77	87	95	111
	117	101	85	55	65	63	37	21	5	
		121	79	71	41	39	75	1		
			19	31	33	99	119			
				15	25	115				
					9					

If the treatment is with the squares of eleven and thirteen, do the same until the above situation is attained.⁴⁶ Then you write the small number following the last of the small numbers already placed (thus 19) in the first border on the bottom, in the third cell from the middle (cell) on the left, and opposite to it its complement.⁴⁷ Then you put the following small number (17) in the first border on the left, in the third cell above the middle (cell), and its complement opposite to it on the right. Then you put the following small term (15) in the second border on the bottom, next to the middle cell,⁴⁸ and its complement opposite to it on the top. Then you put the following small term (13) next to the middle cell in the second border on the left, and its complement opposite to it on the right. Then you put the small term reached (11) in the middle cell of the third border on the left, and opposite to it its complement. Then you put the following small term (9) in the middle cell of the third border on the bottom, and opposite to it on the top its complement. Then you put the following small term (7) next to the middle cell in the second border on the top, and opposite to it on the bottom its complement. Then you put

⁴⁶ With 21, 101 (order 11) and 21, 149 (order 13) placed last. Since for orders 11 and 13 there is the same quantity of cells to be filled with the smaller odd numbers outside the inner square — namely nine on each side, see p. 33 — we give just one illustration. (The numbers of cells on each side are each time square numbers for they are the sum of the successive odd numbers.)

⁴⁷ ‘in the third cell’, here and (*mutatis mutandis*) in what follows: counting the initial cell.

⁴⁸ The situation relative to the middle cell being as in the previous cases.

ما يقابله. ثمّ تثبت العدد القليل الذى يتلو ذلك فى الصفّ الأوّل من
الجهة العليا فى البيت الثالث من الاوسط وبازائه من الجهة السفلى ما
يقابله. ثمّ تثبت العدد القليل الذى يتلو ذلك فى الصفّ الأوّل من الجهة
اليمنى فى البيت الثالث من الاوسط وبازائه ما يقابله فى الجهة اليسرى.¹⁶⁵
فاذا فعلت ذلك فقد اثبتت الاعداد الفردية فى هذين المربعين الى آخرها
فى الوسط.

(11) فان كان عملك فى مربعى خمسة عشر وسبعة عشر عملت مثل ذلك
حتى تنتهى الى الصفّ الثالث [كما عملت فى مربع الثلاثة عشر]. ثمّ
تثبت العدد القليل الذى يتلو ذلك فى الصفّ الأوّل من الجهة السفلى فى
البيت الرابع من الاوسط وما يقابله بازائه. ثمّ القليل الذى يتلو ذلك فى
الصفّ الأوّل من الجهة اليسرى <فى البيت الرابع من الاوسط> وبازائه ما
يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثانى من الجهة السفلى فى البيت
الثالث من الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثانى
من الجهة اليسرى فى البيت الثالث من الاوسط وبازائه ما يقابله. ثمّ الذى
يتلو ذلك فى الصفّ الثالث من الجهة السفلى الى جانب البيت الاوسط
وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثالث من الجهة اليسرى
الى جانب البيت الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ
الرابع من الجهة اليسرى فى البيت الاوسط وبازائه ما يقابله. ثمّ الذى يتلو
ذلك فى البيت الاوسط من الصفّ الرابع من الجهة السفلى وبازائه ما¹⁷⁰

فى اليسرى ما يقابله [ما يقابله فى الجهة اليسرى 165 || A الوسط [الاوسط 165 || A. om. [العدد
D. اللب [الوسط 167 || D على A. عن [الى 166 || D است A. ابنت [اثبتت 166 || D.

(11) A: 12^v, 17-13^r, 20 (ed. ll. 577-599); D: 81^v, 19-82^r, 6.

tit. D : فصل. الثالث :

فى [فى مربع الثلاثة عشر 169 || A فى مربع ١٥ او ١٧ [فى مربعى خمسة عشر وسبعة عشر 168
[القليل 171 || A الوسط [الاوسط 171 || D. om. [القليل 170 || A. om. [العدد 170 || A ١٣
om. [ثمّ الذى ... ما يقابله 174-5 || D يتلوه [يتلو ذلك 171 || D. A, om. post ذلك
فى 178-9 || AD اليسرى [السفلى 176 || A. add. [تثبت sc. سبت [ثمّ 175 || D (homoeotel.)
فى الصفّ الرابع فى البيت الاوسط من الجهة [الصفّ الرابع من الجهة اليسرى فى البيت الاوسط
[فى البيت الاوسط 180 || D فى البيت الاوسط من الصفّ الرابع من الجهة اليسرى A. اليسرى

the following small number (5) in the second border on the right, next to the middle cell, and opposite to it on the left its complement. Then you put the following small number (3) in the first border on the top, in the third cell from the middle (cell), and opposite to it on the bottom its complement. Then you put the following small number (1) in the third cell from the middle (cell) in the first border on the right, and opposite to it on the left its complement. Once you have done that, you will have put the odd numbers, to the last, in the central part of these two squares.⁴⁹

(§ 11. Filling the remainder of the rhomb for the orders 15 and 17)

If you are dealing with the squares of fifteen and seventeen, you do the same until you reach the third border [as you did for the square of thirteen].⁵⁰ Then you put the following small number (27) in the first border on the bottom, in the fourth cell from the middle one, and its complement opposite to it; then the following small number (25) in the first border on the left, in the fourth cell from the middle one, and opposite to it its complement; then the following one (23) in the second border on the bottom, in the third cell from the middle one, and opposite to it its complement; then the following one (21) in the second border on the left, in the third cell from the middle one, and opposite to it its complement; then the following one (19) in the third border on the bottom, next to the middle cell, and opposite to it its complement; then the following one (17) in the third border on the left, next to the middle cell, and opposite to it its complement; then the following one (15) in the fourth border on the left, in the middle cell, and opposite to it its complement; then the following one (13) in the fourth border on the bottom, in the middle cell, and opposite to it its complement; then the following one (11) in the third border on the top, next to the middle cell, and opposite to it its complement; then the following one (9) in the third border on the right, next to the middle cell, and opposite to it its complement; then the following one (7) in the second border on the top, in the third (cell) from the middle one, and opposite to it its complement; then the following one (5) in the second border on the right, in the third (cell) from the middle one, and opposite to it its complement; then the following one (3) in the first border on the top, in the fourth (cell) from the middle one, and opposite to it its complement; then the following one (1) in the first border on the right, in the fourth (cell) from the middle one, and opposite

⁴⁹ 'in the central part': meaning, as in § 7 (p. 31), in the inner (rhombic) part.

⁵⁰ We shall have placed the pairs up to and including 29, 197 and 29, 261 (29 taking the place of 1 in the previous square). From what follows can be inferred the general way of filling the lateral triangles in the rhomb.

يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثالث من الجهة العليا الى جانب البيت الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثالث من الجهة اليمنى الى جانب البيت الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثانى من الجهة العليا فى الثالث من الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الثانى من الجهة اليمنى فى الثالث من 185 الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الاول من الجهة العليا فى الرابع من الاوسط وبازائه ما يقابله. ثمّ الذى يتلو ذلك فى الصفّ الاول من الجهة اليمنى فى الرابع من الاوسط وبازائه ما يقابله. فاذا فعلت ذلك فقد أتيت على الافراد على ما اردت فى هذين المربعين. فاذا اردت ما فوقهما نحوّت هذا النحو من الاثبات فى التدرّج. 190

(12) وتجد حينئذٍ المربعات تتقسم اقسامًا يحتاج كلّ قسم منها فى ترتيب الاعداد الزوجيّة فى الزوايا الى عمل خلاف عمل الآخر. فتكون التسعة والثلاثة عشر والسبعة عشر وما زاد عليها من الاعداد اربعة اربعة [فى قسم] والسبعة والاحد عشر والخمسة عشر والتسعة عشر وما زاد عليها 195 من الاعداد اربعة اربعة [فى قسم]. وفى مربّع الخمسة فقط عمل ليس يجوز فيه <على ما فى> غيره [من هذا الباب].

ثمّ الذى ... ما 181-2 || الجهة اليسرى (...) فى البيت الاوسط. *et iter.* منه فى *A* *postea hab.* على 189 || *om. (homoeotel.) D* || ثمّ الذى ... ما يقابله 183-5 || *A* *om. (homoeotel.)* يقابله 190 || *AD* فوقها [فوقهما 190 || *D* فان [فاذا 190 || *D* اوردا [اردت 189 || *D* *om. (pr.)* فى الاثبات فى *A*, *om. D*.

(12) *A*: 13^r, 20 - 13^v, 3 (ed. ll. 600-603); *D*: 82^r, 9 - 82^r, 11.

الباب الثالث فى اثبات الازواج فى زوايا المربّع الكبير خارج اللب : *tit. D*

D على قسمين [اقسامًا 191 || *A* تقسم [تتقسم 191 || *AD* المربعات حينئذٍ [حينئذٍ المربعات 191 التسعة والثلاثة عشر 192-3 || *D* *add.* الخمسة و [فتكون 192 || *D* العمل الآخر [عمل الآخر 192 || 193-4 || *A* ٤ ٤ [اربعة اربعة 193 || *AD* الافراد [الاعداد 193 || *A* ٩ ١٣ و ١٧ [والسبعة عشر 195 || *D* *om.* [والتسعة عشر 194 || *A* [والخمسة عشر 194 || *om. A* [فى قسم *inven.* [وفى مربّع ... هذا الباب 195-6 || *A* ٤ ٤ [اربعة اربعة 195 || *D* *om.* [من الاعداد *AD* فى [من 196 || *D* منه [فيه 196 || *A* ٥ [الخمسة 195 || *AD* (l. 202) ستة عشر *infra post*

to it its complement. Once you have done that, you will have finished with the odd (numbers) for these two squares in the desired way.

							213							
						207	189	11						
					203	183	173	35	7					
				199	179	167	165	51	31	3				
			25	79	87	83	153	155	159	75	201			
		21	45	85	99	103	133	135	95	141	181	205		
	17	41	57	81	101	111	109	119	125	145	169	185	209	
15	39	55	63	77	97	121	113	105	129	149	163	171	187	211
	217	193	177	157	137	107	117	115	89	69	49	33	9	
		221	197	161	131	123	93	91	127	65	29	5		
			225	151	139	143	73	71	67	147	1			
				27	47	59	61	175	195	223				
					23	43	53	191	219					
						19	37	215						
							13							

If you wish (to fill squares) above these two, continue placing (the numbers) step by step in the same way.⁵¹

(Placing the even numbers)

(§ 12. Different treatments for placing the even numbers)⁵²

At this point you will find that the squares are divided into classes requiring each, for the arrangement of the even numbers in the corner (triangles), a treatment distinct from that of the other.⁵³ There are (the squares of) nine, thirteen, seventeen, and those larger numbers (resulting from adding) successively 4 [in class]; and (the squares of) seven, eleven, fifteen, nineteen, and those larger numbers (resulting from adding) successively 4 [in class]. Only for treating the square of five one cannot proceed as for the others [of this section].⁵⁴

⁵¹ Surrounding with the next numbers the rhomb's sides filled previously.

⁵² Namely for $n = 5$ (isolated case), $n = 4t + 1$ ($t > 1$), $n = 4t + 3$.

⁵³ Meaning: the treatment of one is different from that of its immediate neighbours (this will be explained now). An early reader pointed out what these classes are.

⁵⁴ This last sentence is found between § 13 and § 14 in the two MSS.

(13) فامّا مربّع الخمسة وما كان من نوعها فانه يبقى من الصفّ الاول المحيط بالمربّع الداخل اربعة ابيات الزوايا وثمانية ابيات تتصل بها من كلّ جانب بيتان فارغة كلّها. ويبقى من «الصفّ» الثاني ابيات الزوايا واربعة وعشرون بيتاً تتصل بها فارغة كلّها. ومن الصفّ الثالث ابيات الزوايا واربعون بيتاً تتصل بها فارغة كلّها. وكذلك ابداً كلّ صفّ يزيد على الذي قبله بعد ابيات الزوايا ستة عشر.

(14) «فامّا مربّع الخمسة فان الصفّ الاعلى اذا اثبت الاعداد الفردية فيه يزيد على حقّه اثني عشر وينقص الاسفل عنه اثني عشر والصفّ الايمن منه يزيد على حقّه عشرة وينقص الايسر عنه عشرة.» فامّا سائر المربعات التي تشاركه [في قسم] فان الصفّ الاول الاعلى من مربّع تسعة اذا اثبت الاعداد الفردية فيه على هذا الترتيب يزيد على حقّه [الواجب له] عشرين وينقص الاسفل «عنه» عشرين والصفّ الاول الايمن منه يزيد على حقّه ثمانية عشر وينقص الايسر عنه ثمانية عشر. ويزيد الثاني الاعلى ستة وثلاثين وينقص الاسفل مثلها ويزيد الثاني الايمن اربعة وثلاثين وينقص الايسر مثلها. وفي مربّع الثلاثة عشر يزيد الاول الاعلى ثمانية

(13) A: 13^v, 3-13^v, 9 (ed. ll. 604-609); D: 82^r, 12-82^r, 16.

tit. D : الاول . فصل

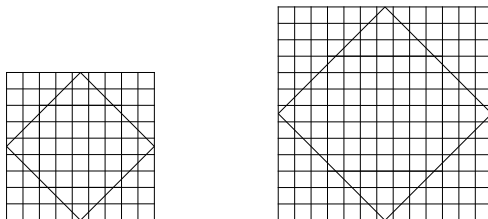
[واربعة وعشرون بيتاً 200 || A ثمانية 198 || D طريقها 197 || نوعها 197 || AD رسم 197
202 || D زيد 201 || A om.] ومن الصفّ ... فارغة كلّها 200-1 || A واربعة واسين ستان
A. (sic) ٢٤] ستة عشر 202 || D بعده A بعدد] بعد

(14) A: 13^v, 9-13^v, 21 (ed. ll. 610-623); D: 82^r, 16-82^v, 1.

206 || A في الصفّ] فان الصفّ 206 || D om.] في قسم 206 || A سار corr. ex 206
الايمن [الاول الايمن 208 || D om.] فيه على 207 || D رنتت A ابنت] اثبت 207 || A ٩] تسعة
om.] على حقّه 209 || D زيد A, (sæpissime) يزيد pro (زيد) ترد منه] منه يزيد 208 || A الاول
209 || A ١٨] ثمانية عشر 209 || D om.] عنه 209 || A ١٨] ثمانية عشر 209 || D
٣٤] اربعة وثلاثين 210 || D وزيد] ويزيد 210 || A (sic) ١٦] ستة وثلاثين 210 || D وزيد] ويزيد
ثمانية 211-2 || A ١٣] الثلاثة عشر 211 || D om.] مربّع 211 || A الاسفل] الايسر 211 || A

(§ 13. Quantity of cells remaining empty in squares of orders $n = 4t + 1$)

For the square of five and those of the same kind (the situation is the following).⁵⁵



There remain as empty cells in the first border — which surrounds the inner square — the four corner cells and eight cells adjacent to them, two on each side;⁵⁶ in the second border, there remain as empty cells those at the corners and twenty-four cells adjacent to them; in the third border, the empty cells are those at the corners and forty cells adjacent to them. It will always be like that: each border has, excepting the corner cells, sixteen more (empty cells) than the preceding border.⁵⁷

	N_1	N_2	N_3	N_4	N_5
$(n = 4t + 1)$	$4 + 8$	$4 + 24$	$4 + 40$	$4 + 56$	$4 + 72$

(§ 14. Excesses and deficits in each row for the order $n = 4t + 1$)

⟨ For the square of five, after placing the odd numbers in it, the upper row is in excess over its sum due by 12, the lower one is in deficit by 12, the right-hand row is in excess over its sum due by 10, and the left-hand one is in deficit by 10. ⟩⁵⁸ For the other squares associated with it [in class] (the situation is the following). After arranging the odd numbers in the above manner, the first upper row of the square of nine is in excess over its sum due [required for it] by 20, and the lower one is in deficit by 20;⁵⁹ the first right-hand row is in excess over its sum due by 18, and the left-hand one is in deficit by 18. The second upper (row) is in excess by 36, and the lower one is in deficit by the same (amount); the second

⁵⁵ The square of order 5 belongs to the first class for determining the number of still empty cells in the border, but not for filling them. Its filling will be explained separately in § 23.

⁵⁶ Meaning: on each of the four sides of the border (not: on each side of a corner cell).

⁵⁷ In our table, N_i is the whole quantity of cells remaining empty in the i th border (counted from the central square already filled).

⁵⁸ We deemed it necessary to add this sentence, not appearing in the two MSS.

⁵⁹ ‘required for it’ must be an early reader’s addition: the concept of ‘sum due’ will be explained below.

وعشرين وينقص الاسفل مثلها ويزيد الاول الايمن ستة وعشرين وينقص
الايسر مثلها. ويزيد الثاني الاعلى اثنين وخمسين وينقص الاسفل مثلها
ويزيد الثاني الايمن خمسين وينقص الايسر مثلها. ويزيد الثالث الاعلى
ستة وسبعين وينقص الاسفل مثلها ويزيد الثالث الايمن اربعة وسبعين
وينقص الايسر مثلها. وكذلك سائرهما.

وتكون التسعة ي زيد الثاني على الذى قبله ستة عشر. والثلاثة عشر
يزيد الاول على اول التسعة ثمانية ثم كل صف على الذى قبله اربعة
وعشرين. والسبعة عشر ي زيد الاول على اول الثلاثة عشر ثمانية ثم ي زيد
كل صف على الذى قبله اثنين وثلاثين. ثم كذلك ابدا كل صف على الذى
قبله ثمانية.

(15) ومعنى زيادته على حقه ان الذى يجب للبيت بقسطه [من جملة
العدد] مثل عدد الواسطة [فان كل بيت من ابيات مرتع خمسة يجب له
بحقه ثلاثة عشر] فاذا اجملت ما يقع فى كل صف من الافراد ثم قسمته
على عدد الابیات التى وقع ذلك فيها فقصر القسم عن الواسطة فقد وقع
فى الابیات اقل من حقا بمقدار ذلك التقصير مضروبا فى الابیات وان
زاد فالزيادة على حسب ذلك. فيحتاج ان يكون ترتيب الاعداد الزوجية
[بعد ذلك] فيما بقى فى كل صف من الابیات الفارغة ترتيبا يفى بمقدار

A ٥٠ || خمسين 214 || A ٥٢ || اثنين وخمسين 213 || A ٢٦ || ستة وعشرين 212 || A ٢٨ || وعشرين
تكون التسعة [وتكون التسعة 217 || A ٧٤ || اربعة وسبعين 215 || A ٧٦ || ستة وسبعين 215 ||
218-9 || A ٨ || ثمانية 218 || A والثلاثة [والثلاثة عشر 217 || A ١٦ || ستة عشر 217 || A, om. D
[اول 219 || D ريد (pr.) ي زيد 219 || A ١٧ || والسبعة عشر 219 || A ٢٤ || اربعة وعشرين
يزيد 219 || A ٨ || ثمانية 219 || A om. [الثلاثة عشر 219 || A pr. scr. et del. (الى) AD الاول
(post.) om. A 220 || اثنين وثلاثين 220 || A (sic) ٣٣ || ثمانية 221 || A.

(15) A: 13^v, 21-14^r, 9 (ed. ll. 624-632); D: 82^v, 1-82^v, 6.

A, وان كل [فان كل 223 || D om. [مثل عدد 223 || AD البيت [للبيت 222 || D om. [ان 222
|| D بلث عشر, A ١٣ || ثلثة عشر 224 || A ٥ || خمسة 223 || A الابیات [ابیات 223 || D فان كان
226 || D من [فى 226 || D مضروب [مضروبا 226 || D على [عن 225 || A جملة [اجملت 224
بمقدار 229 || AD بنقصان [بمقدار النقصان 228-9 || A فعلى [فالزيادة على 227 || D فان [وان

right-hand (row) is in excess by 34, and the left-hand one is in deficit by the same (amount). For the square of thirteen, the first upper (row) is in excess by 28, and the lower one is in deficit by the same (amount); the first right-hand (row) is in excess by 26, and the left-hand one is in deficit by the same (amount). The second upper (row) is in excess by 52, and the lower one is in deficit by the same (amount); the second right-hand (row) is in excess by 50, and the left-hand one is in deficit by the same (amount). The third upper (row) is in excess by 76, and the lower one is in deficit by the same (amount); the third right-hand (row) is in excess by 74, and the left-hand one is in deficit by the same (amount). Likewise for the others.⁶⁰

	Δ_1^L	Δ_1^C	Δ_2^L	Δ_2^C	Δ_3^L	Δ_3^C	Δ_4^L	Δ_4^C	Δ_5^L	Δ_5^C
$n = 5$	12	10								
$n = 9$	20	18	36	34						
$n = 13$	28	26	52	50	76	74				
$n = 17$	36	34	68	66	100	98	132	130		
$n = 21$	44	42	84	82	124	122	164	162	204	202

(In the square of) nine, the second (row) has 16 more than the (row) before. (In the square of) thirteen, the first (row) has 8 more than the first (of the square) of nine, then each row has 24 more than the (row) before. (In the square of) seventeen, the first (row) has 8 more than the first of (the square of) thirteen, then each row 32 more than the (row) before. Then, in the same manner always, each row will have 8 more than the one before.⁶¹

(§ 15. The concept of ‘sum due’)

‘Excess over its sum due’ means (the following). The proper required (amount) for a cell [of the totality of the numbers]⁶² equals the quantity of the median [thus each cell of the square of five has a sum due of 13]⁶³. Therefore you will add the odd numbers found in each row and divide the (result) by the number of cells concerned; if the quotient is less than the median, the cells will contain less than their sum due by the result of multiplying this deficit by the (number of) cells (filled); accordingly

⁶⁰ See our table, with Δ_i^L the excesses, or deficits, in the horizontal rows (‘Lines’) of the i th border and Δ_i^C those in the corresponding vertical rows (‘Columns’).

⁶¹ This is true for the first border. For the others, it will be successive multiples of 8.

⁶² Early (now misplaced) gloss explaining ‘median’ (thus $\frac{n^2+1}{2}$ for a square of order n); this term needs no explanation since it has been encountered regularly from § 3 on.

⁶³ Banal, and known from § 4.

النقصان ان كان وينقص بمقدار الزيادة ان كانت. وسنبيّن *«العمل في»* ذلك في موضعه عند تفسير أمر الأزواج.

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(16) وأما قسم السبعة وما كان من نوعها فإنه يبقى من الصفّ الأوّل المحيط بالمرّبع الداخل أربعة ابيات الزوايا فقط فارغة. ومن الصفّ الثاني ابيات الزوايا وستّة عشر بيتًا تتّصل بها من كلّ جانب أربعة. ومن الصفّ الثالث ابيات الزوايا واثنان وثلثون بيتًا تتّصل بها فارغة. وكذلك *«أبدًا»* كلّ صفّ يزيد على الذي قبله بعد ابيات الزوايا ستّة عشر.

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(17) ويكون الصفّ الأوّل الاعلى من كلّ المربّعات في هذا النوع يزيد على حقّه أربعة وينقص الاسفل مثلها ويزيد الأوّل الايمن على حقّه اثنين وينقص الايسر مثلها. ويكون الصفّ الثاني الاعلى من السبعة يزيد على حقّه عشرين والثاني الايمن ثمانية عشر. ويزيد الثاني الاعلى في مرّبع احد عشر على حقّه ثمانية وعشرين والثاني الايمن ستّة وعشرين والثالث الاعلى اثنين وخمسين والثالث الايمن خمسين وينقص ما يقابل جميع ذلك مثل الزيادة. والخمسة عشر يزيد الثاني الاعلى على الأوّل اثنين وثلثين

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ان [الأزواج 230 || A تبين [تفسير 230 || D وسنبيّن 229 || AD بمقدار زيادة [الزيادة add. D. شا الله

(16) A: 14^r, 9-14^r, 14 (ed. ll. 633-637); D: 82^v, 7-82^v, 10.

tit. D : فصل. الثاني :

231 || A الزا corr. ex [الزوايا 232 || A ٤ [أربعة 232 || D طريقها [نوعها 231 || AD رسم [قسم 231 || A ٣٢ [واثنان وثلثون 234 || A ٤ [أربعة 233 || A ١٦ [وستّة عشر 233 || A om. [فقط 232 || A. ١٦ [ستّة عشر 235 || A بعده [بعد 235

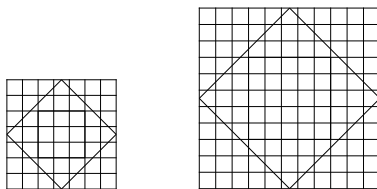
(17) A: 14^r, 14-14^r, 20 (ed. ll. 638-644); D: 82^v, 10-82^v, 16.

238 || D add. من [وينقص 237 || A ٤ [أربعة 237 || D الجنس [النوع 236 || A om. [كلّ 236 || A ٢٨ (sic) [ثمانية عشر 239 || A ٢٠ [عشرين 239 || A ملها [مثلها 238 || A ٢ [اثنين ثمانية 240 || A ٧ (sic) [احد عشر 240 || A in marg. [ويزيد (...) ستّة وعشرين 239-40 || A ٥٠ [خمسين 241 || A ٥٢ [اثنين وخمسين 241 || A ٢٦ [ستّة وعشرين 240 || A ٢٨ [وعشرين 241 || A ٣٢ [اثنين وثلثين 242 || A add. in marg. [جميع 241 || A add. et del. [جميع [وينقص 241 || A ثمانية 243 || A ٣٢ [اثنين وثلثين 243 || D الباقي [الثاني 243 || AD والذي بعده [والثالث 243 || A ٨ ٨ [ثمانية

for an excess if (the quotient) is more.⁶⁴ So the arrangement of the even numbers [after that] in the empty cells of each row must be such that it compensates the amount of the deficit, if any, or falls short by the amount of the excess, if any. We shall show how to do this in the appropriate place, when discussing the question of the even (numbers).⁶⁵

(§ 16. Quantity of cells remaining empty in squares of orders $n = 4t + 3$)

For the class of seven and those of the same kind (the situation is as follows).



There remain as empty cells in the first border — which surrounds the inner square — only the four corner cells; in the second border, the corner cells and sixteen cells adjacent to them, four on each side; in the third border, the empty cells are the corner cells and thirty-two cells adjacent to them. It will always be like that: each border has, excepting the corner cells, sixteen more (empty cells) than the preceding border.

$(n = 4t + 3)$	N_1	N_2	N_3	N_4	N_5
	4	$4 + 16$	$4 + 32$	$4 + 48$	$4 + 64$

(§ 17. Excesses and deficits in each row for the order $n = 4t + 3$)

In all the squares of this kind, the first upper row has an excess of 4 over its sum due, and the lower one has a deficit of the same (amount); the first right-hand (row) has an excess of 2 over its sum due, and the left-hand (row) has a deficit of the same (amount). The second upper row of the (square of) seven has an excess of 20 over its sum due, and the second right-hand (row an excess of) 18. In the square of eleven, the second upper (row) has an excess of 28 over its sum due, the second right-hand (row an excess of) 26, the third upper (row an excess of) 52, the third right-hand (row an excess of) 50, and all their opposite (rows) have a deficit equal to the excess. (Square of) fifteen: the second upper (row)

⁶⁴ This is confusing since the effect of the previous division is cancelled by the subsequent multiplication. To find the excess or deficit displayed by a border's row, just take the difference between the sum found in the m cells already filled in this row and the product of m by the median number.

⁶⁵ §§ 18–22 (equalization rules).

والثالث على الثانى اثنين وثلثين. وكذلك سائرهما يزيد ثمانية ثمانية.

(18) فنحتاج الى ان يكون تأليف الازواج فيها تأليفاً يعدّها والى ان
نعلم كيف المأتى لنطلب الاعداد التى تزيد بمقدار ما يحتاج اليه من
الزيادة او تنقص بمقدار النقصان.

ووجه معرفة ذلك أنّك تؤلف الاثنين والطرف الاخير من الازواج ثمّ
الاربعة وما يقابلها ثمّ كذلك حتى تنتهى الى الطرفين اللذين هما واسطتا
تلك الاعداد الازواج.

(19) فانّك اذا فعلت ذلك وجدت اخير الاعداد القليلة والذى قبله اذا
صيّرا في جهة وصيّر بازائهما ما يقابلهما قصّرت الجهة التى فيها القليلان
عن حقّها باربعة وزادت الاخرى اربعة. فاذا صيّر العددان اللذان
يتلوانهما من القليلة في جهة وبازائهما مقابلهما قصّرت الجهة التى فيها
القليلان عن حقّها باثنى عشر وزادت الاخرى مثلها. واذا فعلت مثل ذلك
بالذين يتلوان ذينك كانت الزيادة والنقصان عشرين. فكذاك ابداً حتى
تنتهى الى الاربعة والاثنين.

[فانّك تجد كلّ عدد من يقصّران عن اللذين قبلهما ثمانية ثمانية فيكون
الاولان يقصّران باربعة واللذان يتلوانهما يقصّران باثنى عشر واللذان

(18) A: 14^r, 20-14^v, 3 (ed. ll. 645-650); D: 82^v, 16-82^v, 19.

tit. (hab. ante معرفة ووجه معرفة infra) D : الفصل. الثالث :

[او 246 || D لطلب A لتطلب] لنطلب 245 || D البانى A البانى [المأتى 245 || A ان] الى ان 244
[يقابلها 248 || A توفق] تؤلف 247 || A ان [انّك 247 || D ومعرفة] ووجه معرفة 247 || AD و
[الاعداد 249 || D وما يقابله] كذلك 248 || A add. مثلها

(19) A: 14^v, 3-14^v, 13 (ed. ll. 651-661); D: 82^v, 19-83^r, 4.

251 || D صيّر صيّرا 251 || D آخر [اخير 250 || A وخبرت] وجدت 250 || D om. [ذلك 250
|| A يقابلها] يقابلها 251 || A بازائهما [بازائهما 251 || D add. in marg.] وصيّر 251 || D الى [فى
|| A الاخيرة] الاخرى 252 || D اربعة A ٤ [اربعة 252 || A تقابلها الفلّيلين] فيها القليلان 251
|| 253 || A om. [يتلوانهما ... باربعة واللذان 253-8 || A واذا] فاذا 252 || A ٤ [اربعة 252
|| D والاثنين 256 || D resc.] عشرين 255 || D اثنى عشر [باثنى عشر 254 || D يقابلها] مقابلهما
يتلوانهما [بعدهما 259 || A ١٢ ١٢] باثنى عشر 258 || D نقصيران [يقصّران 257 || D واللسن

exceeds the first (upper row) by 32, the third the second by 32. And so on for the others, (with excesses and deficits) increasing each time by 8.⁶⁶

	Δ_1^L	Δ_1^C	Δ_2^L	Δ_2^C	Δ_3^L	Δ_3^C	Δ_4^L	Δ_4^C	Δ_5^L	Δ_5^C
$n = 7$	4	2	20	18						
$n = 11$	4	2	28	26	52	50				
$n = 15$	4	2	36	34	68	66	100	98		
$n = 19$	4	2	44	42	84	82	124	122	164	162

(§ 18. Grouping the even numbers by pairs)

Thus we need to write the even (numbers) in the rows so as to equalize them, and (therefore we need) to know how to proceed in searching for the numbers displaying the required amount of the excess or displaying the (required) amount of the deficit.

The way to determine this is (first) to associate 2 and the last even term, then 4 and its complement, and so on until you reach the two terms which are the two medians of these even numbers.⁶⁷

$$\begin{array}{ccccccc}
 2 & 4 & | \dots | & \frac{n^2+3}{2} - 4j & \frac{n^2+7}{2} - 4j & | \dots | & \frac{n^2-5}{2} & \frac{n^2-1}{2} \\
 n^2 - 1 & n^2 - 3 & | \dots | & \frac{n^2-1}{2} + 4j & \frac{n^2-5}{2} + 4j & | \dots | & \frac{n^2+7}{2} & \frac{n^2+3}{2}
 \end{array}$$

(§ 19. Effect of placing a pair of small even numbers in the same row)

Once you have done that, you will find that if the last small number and the preceding one are placed on one side,⁶⁸ and opposite to them their complements, the side containing the two small numbers will be less than its sum due by 4, and the other more by 4. Placing the next two small numbers on one side and opposite to them their complements, the side containing the two small numbers will be less than its sum due by 12, and the other more by the same (amount). Doing the same with the next pair, excess and deficit will be 20. And so on always until you reach 4 and 2.⁶⁹

⁶⁶ The differences of the excesses, both between successive borders and successive squares, are successive multiples of 8. As above (§ 14, *n*. 61), some further explanation might be missing.

⁶⁷ As seen from the table, the pairs of consecutive smaller numbers are of the general form $\frac{n^2+3}{2} - 4j$, $\frac{n^2+7}{2} - 4j$, with j taking successive natural values ($j = 1$ making the last pair $\frac{n^2-5}{2}$, $\frac{n^2-1}{2}$).

⁶⁸ As seen above, the 'last small (even) number' is $\frac{n^2-1}{2}$ and the one before, $\frac{n^2-5}{2}$.

⁶⁹ See the figure below, with the pairs written in their general form (§ 18), and from right to left, as appropriate here.

بعدهما بعشرين ثمّ ثمانية وعشرين ثمّ ستّة وثلاثين ثمّ أربعة واربعين ثمّ
 260 اثنين وخمسين ثمّ ستّين وكذلك ابداً الى الاربعة والاثنين.
 [وانت تجد الصفوف التى تحتاج الى ترتيب الازواج فيها انما تزيد او
 تنقص فى جهتين من جهاتها على هذا الترتيب اما اربعة واما اثني عشر
 واما عشرين واما ثمانية وعشرين واما ستّة وثلاثين وكذلك ابداً على هذا
 المثال. فاذا اقيمت الزيادة بهذه الازواج مقام النقصان (ورسمت بازاء كلّ
 265 عدد منها ما يقابله من الكثيرة) فعُدلت الصفوف من جهتين وبقيت
 جهتان تحتاج الى تعديلهما فتعديلهما بما يبقى من الازواج.]

(20) فاذا أخذت اربعة اعداد من القليلة على النسق واثبتت الاول
 منها والاخير فى جهة والاولى فى الجهة الاخرى التى تقابلها واثبتت
 بازاء كلّ واحد منها ما يقابله من الكثيرة فعُدلت احدى الجهتين
 270 بالاخري.]

[فاذا أخذت اربعة اعداد من القليلة ينقص الاول والثانى منها اذا
 جُمعا عدداً ما وينقص الثالث والرابع اذا جُمعا اقلّ من ذلك العدد بثمانية
 فاذا جمعت الاول والرابع فوجدتهما بنقصان نصف النقصانين مجموعين
 وكذلك اذا جمعت الثانى والثالث نقصا مثل ذلك.]

ثمّ [ثمّ ستّة وثلاثين 259 || A ٢٨ [ثمانية وعشرين 259 || A ٢٠ ٢٠ [بعشرين 259 || A يقصران
 ثمّ 259-60 || D واربعه واربعين A, ثمّ ٤٤ [ثمّ اربعة واربعين 259 || D وستة وثلاثين A, ٣٦
 [جهاتها 262 || D وستين A, ثمّ ٦٠ [ثمّ ستّين 260 || D واثنين وخمسين A, ثمّ ٥٢ [اثنين وخمسين
 || D او [quater] واما 262-3 || D اربعة اربعة A, ٤ [اربعة 262 || A (جهة corr. ex) جهاتها
 [ستّة وثلاثين 263 || A ٢٨ [ثمانية وعشرين 263 || A ٢٠ [عشرين 263 || A ١٢ [اثني عشر 262
 266 || D تعدل [فعُدلت 265 || A الرد corr. ex [الزيادة 264 || D om. [هذا 263 || A ٣٦
 266 || D فتعدّلها A, melius resc. in marg. D. [تعدّلها 266 || A تعديلها [تعدّلها 266 || A تعديلها

(20) A: 14^v, 13-14^v, 20 (ed. ll. 662-669); D: 83^r, 4-83^r, 9.

AD جمعت [فاذا جمعت 273 || A سمنيه [بثمانية 272 || A om. [العدد 272 || D واذا [فاذا 271
 [مجموعين 273 || D النقصان corr. ut vid. ex [النقصانين 273 || D وجدتهما [فوجدتهما 273 ||
 [واللذان 276 || A وماله لك ان [ومثال ذلك ان 275 || A om. [نقصا 274 || A مجموعهما corr. ex

	$\frac{n^2+7}{2} - 4j$	$\frac{n^2+3}{2} - 4j$		$\Rightarrow n^2 + 1 + 4 - 8j$
	$\frac{n^2-5}{2} + 4j$	$\frac{n^2-1}{2} + 4j$		$\Rightarrow n^2 + 1 - 4 + 8j$

[Thus you find that (the sum of) each pair of (small) numbers is less than that before⁷⁰ by successive (additions of) 8: the first pair is in deficit by 4, the next pair is in deficit by 12, the subsequent pair by 20, then successively (by) 28, 36, 44, 52, 60, and so on always till 4 and 2.]⁷¹

[Now you find that the rows which require arranging even (numbers) in them have indeed, on pairs of (opposite) sides, this succession of excesses or deficits, namely 4, 12, 20, 28, 36 and so on always in this way.⁷² Thus, placing the excess of these even (numbers) where the deficit is (and writing opposite to each (placed) number its complement among the large (numbers)), you will have equalized the (horizontal) rows on two sides, and you will be left with equalizing two (vertical) sides, which you will do with the remaining even numbers.]⁷³

(§20. ‘Neutral placings’ by means of two consecutive pairs)⁷⁴

⟨If you take four (even) consecutive small numbers and put the first and the last on one side and the two middle on the other, opposite side, and put opposite to each number the large one which is its complement, you will have equalized these two sides.⟩⁷⁵

	$\alpha + 6$	$n^2 + 1 - (\alpha + 4)$	$n^2 + 1 - (\alpha + 2)$	α		$\Rightarrow 2(n^2 + 1)$
	$n^2 + 1 - (\alpha + 6)$	$\alpha + 4$	$\alpha + 2$	$n^2 + 1 - \alpha$		$\Rightarrow 2(n^2 + 1)$

[If you take four small (even, consecutive) numbers, the deficit of the

⁷⁰ Rather: than the first (largest) pair.

⁷¹ Early reader’s addition: it adds nothing and merely extends the list of differences. It is true that the original text as preserved did not mention that the differences increase by successive additions of 8.

⁷² As seen in §§14 and 17 for the horizontal rows.

⁷³ Early reader’s addition, perhaps the same reader as before. First, we are here in the domain of general equalization rules; their application will follow (§§24–35). Furthermore, the alleged equalization is not applicable as such since equalization of two opposite horizontal rows depends on the quantities put beforehand in their end cells.

⁷⁴ *Neutral placings* provide exactly the sum due for four cells.

⁷⁵ This sentence is missing. The two subsequent interpolations must be some attempt to restore the meaning.

275 [ومثال ذلك ان العددين الاخيرين من القليلة اذا جُمعا قصّرا عن
حقّهما باربعة واللذان قبلهما يقصّران باثنى عشر فاذا جمعت الاخير
والذى قبله بعددين قصّرا بثمانية وكذلك اذا جمعت الذى قبل الاخير
والذى يليه قصّرا بثمانية وهو مثل نصف الاثنى عشر والاربعة مجموعين].
وهذا ايضاً ممّا يحتاج الى معرفته لآنك تستعمله كثيراً.

280 (21) واذا رسمت العدد الاول القليل فى جهة او اثنى عدد شئت من
الليلة شئت رسمت الذى يتلوه فى الجهة الاخرى التى تقابلها ورسمت بازاء
كل واحد منهما ما يقابله من الكثيرة قصّرت الجهة التى رسمت فيها القليل
الاول عن حقّها باثنين وزادت الاخرى اثنين. فان رسمت عدداً من القليلة
فى جهة ورسمت فى الجهة التى تقابلها العدد القليل الثالث من ذلك العدد
285 قصّرت الجهة التى فيها القليل الاول عن حقّها باربعة. وكذلك كلّما باعدت
بينهما بعدد زاد نقصان اثنين ابداً.

[ومثال ذلك اّنك اذا اثبتت الاربعة فى جهة واثبتت فى الجهة التى
تقابلها ستة وبازاء كلّ واحد منهما ما يقابله نقصت الجهة التى فيها
الاربعة عن حقّها اثنين وزادت الجهة التى فيها الستة اثنين. فان رسمت
290 مكان الستة ثمانية وفعلت مثل ذلك نقصت جهة الاربعة اربعة وزادت
جهة الثمانية اربعة. وكذلك <ابداً> كلّما باعدت بينهما فعلى هذا قدر
زيادتهما].

وذلك [وهو مثل 278 || *A* om. بعددين ... والذى يليه 277-8 || *D* قبلهما] قبله 277 || *D* والذين
A دائماً [كثيراً 279 || *A* ما ممّا 279 || *A* om. ايضاً 279 || *D* مجموعه مجموعه 278 || *D*

(21) *A*: 14^v, 20-15^r, 12 (ed. ll. 670-682); *D*: 83^r, 9-83^r, 17.

283 || *A* باثنين [اثنين 283 || *D* الكبيره [الكثيرة 282 || *A* منها [منهما 282 || *A* اذا [واذا 280
pr. قبلها [فيها 285 || *A* add. الاخرى [فى الجهة 284 || *A* القليل [الليلة 283 || *A* رست [رسمت
iter. in وكذلك (...) ابداً 285-6 || *A* (sic) [اثنين 286 || *A* 4 [باربعة 285 || *A* *scr. et del.*
[اّنك 287 || *D* مثال [ومثال 287 || (sic) 6 *scr.* اثنين *pro vero* هما *scr.* بينهما 287 || *A*, *sed pro*
[واثبتت فى 287 || *A* 4 [الاربعة 287 || *D* است, (saepius) *A* ابت (pr.) اثبتت 287 || *D* om.
289 || *A* 4 [الاربعة 289 || *A* 6 ستة 288 || *A* add. الآخري [الجهة 287 || *D* وفى *A*, ايتيت فى
290 || *A* 8 [ثمانية 290 || *A* 6 [الستة 290 || *A* 2 [اثنين 289 || *A* 6 [الستة 289 || *A* 2 (pr.) اثنين
|| *A* 8 [الثمانية 291 || *D* حقّه *corr. ut vid. ex* جهة 291 || *A* 4 [اربعة 290 || *A* 4 [الاربعة
AD. زيادتها [زيادتهما 292 || *A* 4 [اربعة 291

sum of the first and the second is a certain number, and the deficit of the sum of the third and the fourth is this number less 8.⁷⁶ Then adding the first and the fourth you will find that they have a deficit of half the sum of the two deficits; likewise, adding the second and the third gives the same deficit.⁷⁷

[For instance, the sum of the last two small numbers is less than their sum due by 4, and (the sum of) the two preceding numbers is less by 12;⁷⁸ then, adding the last and that separated by a pair will make a deficit of 8, and, likewise, adding the one preceding the last one and the next one will make a deficit of 8;⁷⁹ now this equals half the sum of 12 and 4.]

The knowledge of this is again necessary, for you will use it frequently.⁸⁰

(§ 21. Effect of placing two small even numbers in opposite rows)

When you write the first small number,⁸¹ or any small number, on one side and then the subsequent number on the other, opposite side, and you write the two large numbers which are their complements opposite to them, the side where you have written the first small number will be less than its sum due by 2, whereas the other (side) will be in excess by 2. If you write some small number on one side and you write on the opposite side the third small (even) number (counted) from this one, the side containing the first small number will be less than its sum due by 4. And so on: whenever you increase the distance between these two (small numbers) by one number, the deficit is always increased by 2.

	$n^2 + 1 - (\alpha + 2s)$	α		$\Rightarrow n^2 + 1 - 2s$
	$\alpha + 2s$	$n^2 + 1 - \alpha$		$\Rightarrow n^2 + 1 + 2s$

[For instance, if 4 is put on one side and 6 on the opposite side, and opposite to each its complement, the side containing 4 will be less than

⁷⁶ See § 19, in particular interpolations (the same glossator may thus be at work here). Indeed, $n^2 + 1 - [(\alpha + 4) + (\alpha + 6)] = n^2 + 1 - [\alpha + (\alpha + 2)] - 8$.

⁷⁷ The inference being that since placing them on opposite sides makes equal deficits while their complements placed opposite provide the same amount as excess, the two opposite rows will thus be equalized.

⁷⁸ See § 19; indeed, $\frac{n^2-1}{2} + \frac{n^2-5}{2} = n^2 + 1 - 4$, $\frac{n^2-9}{2} + \frac{n^2-13}{2} = n^2 + 1 - 12$.

⁷⁹ $\frac{n^2-1}{2} + \frac{n^2-13}{2} = n^2 + 1 - 8$, $\frac{n^2-5}{2} + \frac{n^2-9}{2} = n^2 + 1 - 8$.

⁸⁰ Use of neutral placings in §§ 31, 33, 34; again, this time for even-order squares and consecutive natural numbers, in §§ 42-43 & 51-54.

⁸¹ We thus now consider the sequence beginning with 2.

(22) وإذا اثبتت عددان من القليلة في زاويتين على نسق واثبتت بازاءهما قطراً ما يقابلهما من الكثيرة واثبتت القليلين في العليا والكثيرين في السفلى فان نقصان العددين القليلين على قدر ترتيبهما ان كانا الاخيرين 295 فاربعة وان كانا اللذين قبلهما فاثنا عشر وكذلك ابداً بزيادة ثمانية ثمانية حتى تنتهى الى الاثنين والاربعة وتكون الجهة اليمنى تزيد على حقها او ينقص عنه اثنين ابداً لا زيادة فيها ولا نقصان. فافهم هذا كله فانه مما تحتاج اليه في تأليف الازواج في هذا النوع.

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[مثال الاعمال في جميع ما ذكرناه]

(23) اما العمل في مربع خمسة في خمسة فهو ان تثبت الاعداد الفردية في مربع ثلثة في ثلثة الذى في وسطه على ما شرحنا فيبقى منها الواحد والخمسة والعشرون والثلثة والثلثة والعشرون. فتثبت الواحد في البيت

(22) A: 15^r, 12-15^r, 17 (ed. ll. 683-688); D: 83^r, 17-83^r, 21.

وإذا اثبت اربعة اعداد في الزوايا كل اثنين منها على نسق فيقابلهما [واذا ... من الكثيرة 293-4 AD فاثبت] واثبتت 294 || [يقابلهما scr. D فيقابلهما hab. A ٢, pro AD طرفهما قطراً على قدرين بينهما على قدر ترتيبهما 295 || A (homoeotel.) om. [في العليا ... القليلين 294-5 D, ١٢ A, فاثنا عشر 296 || D كان 296 || A ٤ فاربعة 296 || resc. in marg.) بينهما D و [والاربعة 297 || D الستة A (sic) ٦ الاثنين 297 || A ٨ ٨ ثمانية ثمانية 296 || D فاني عشر add. in marg. 299 || A ٢ اثنين 298 || D om. تزيد 297 || D ويكون] وتكون 297 || A ٤ من هذا النوع [في هذا النوع 299 || D عند (pr.) في 299 || D om. هذا كله 299 || A, om. D.

(23) A: 15^r, 18-15^v, 8 (ed. ll. 690-699); D: 83^r, 21-83^v, 2.

tit. ad §§ 23-35 hab. A (l. 300, مثال الاعمال في جميع ما ذكرناه)

tit. (ad §§ 23-31) D فصل الرابع :

في 302 || D om. [فهو 301 || A (sic) ٥ ٥ خمسة في خمسة 301 || D والعمل] اما العمل 301 الواحد والخمسة والعشرون والثلثة 302-3 || D add. supra lin. [ما 302 || A om. ثلثة الذى hab. in marg. (الثلثة والثلثة والعشرون sc. ult. part. A; ١ و ٢٥ و ٣ و ٢٣ والثلثة والعشرون

its sum due by 2 and the side containing 6 will be more by 2. If you write 8 instead of 6 and do the same, the side of 4 will be less by 4 and the side of 8 more by 4. And always likewise: whenever you increase the distance between these two, the amount of their augmentation (will vary) accordingly.]⁸²

(§22. Effect of placing pairs of small numbers in corners)

If you place two small (even, consecutive) numbers in consecutive corners and their large complements diagonally opposite, and you put the two small numbers on the top and (therefore) the two large ones on the bottom, the deficit of the two small numbers will depend on their rank: 4 if they are the last two, 12 if they are the two previous ones, and so on always, with regular additions of 8, until you reach 2 and 4.⁸³ (But) the right-hand side will always have, relative to its sum due, an excess of 2, or a deficit of 2,⁸⁴ without any increase and decrease.

$\frac{n^2 + 7}{2} - 4j$		$\frac{n^2 + 3}{2} - 4j$	$\Rightarrow$	$n^2 + 1 + 4 - 8j$
$\frac{n^2 - 1}{2} + 4j$		$\frac{n^2 - 5}{2} + 4j$	$\Rightarrow$	$n^2 + 1 - 4 + 8j$

$\Downarrow$
 $n^2 + 1 + 2$

$\Downarrow$
 $n^2 + 1 - 2$

You must understand all this: it belongs to what you need for writing the even (numbers) in (squares of) this kind.

[Examples of treatments for all that we have explained]⁸⁵

(Square of order 5)

(§23. Filling the square of order 5)

Treatment for the square of five by five. You put the odd numbers in the central three by three square as we have explained.⁸⁶ Those remaining are 1, 25, 3, 23. You put 1 in the lower middle cell and 25 opposite to it

⁸² This example, with the next even numbers after 2, must also be interpolated — if only because we are to remain in the domain of purely theoretical equalization rules.

⁸³ According to the equalization rule of § 19. See the figure, with the smaller pair (in its general form, see § 18) placed at the top (the lesser on the right).

⁸⁴ Depending on whether it contains the larger small number or not.

⁸⁵ Appropriate, though only in MS. $\mathcal{A}$: the equalization rules will now be applied to the two order classes: in §§ 24–31 (orders $n = 4t + 1$, $t > 1$) and §§ 32–35 (orders $n = 4t + 3$); the particular case $n = 5$ will be considered first.

⁸⁶ See § 3 (and § 8, n . 41).

الايوسط الاسفل والخمسة والعشرين بازائه في الجهة العليا وثبتت الثلاثة في
 البيت الاوسط الايسر والثلاثة والعشرين بازائها في الجهة اليمنى. ثم تثبت
 الاثنين في الزاوية العليا اليمنى وبازائهما قطراً في الزاوية اليسرى السفلى
 ما يقابلها وهو اربعة وعشرون. وتثبت الاربعة في الزاوية العليا اليسرى
 وبازائها قطراً في الزاوية اليمنى السفلى ما يقابلها وهو اثنان وعشرون.
 وتثبت الستة في الجهة اليمنى وبازائها في اليسرى ما يقابلها وهو عشرون.
 وتثبت الثمانية والعشرة في الجهة السفلى والاثنى عشر في الجهة اليمنى
 وتثبت بازاء كل واحد منها ما يقابله.

(24) فاذا تجاوزت الخمسة الى التسعة والثلاثة عشر والسبعة عشر وما كان
 من هذا النوع فاثبت آخر الاطراف القليلة من الازواج في الزاوية العليا
 اليسرى من الصف الاول وهو الذى يلي المربع الداخلى الذى حشوته
 بالافراد وبازائه قطراً في الزاوية اليمنى السفلى من الصف الاول ما يقابله.
 واثبت الطرف القليل الذى يليه قبله في الزاوية اليمنى العليا من الصف
 الاول وبازائه قطراً في الزاوية اليسرى السفلى الطرف الذى يقابله. [فاذا

305 || A || ٣ و [وتثبت الثلاثة 304 || D والخمسة والعشرون A, و ٢٥] والخمسة والعشرين 304 || D
 بازائها في 305 || D والثلاثة (والاثنى 11. ex supra lin. corr.) والعشرون A, و ٢٣] والثلاثة والعشرين
 306 || A || ٢ [الاثنين 306 || D بازائه في الجهة اليمنى A, في الجهة اليمنى التى تقابله [الجهة اليمنى
 من الزوج AD; يقابلها [يقابلها 307 || A. ut vid. الفصل [السفلى 306 || AD وبازائها [وبازائهما
 اليسرى العليا [العليا اليسرى 307 || A || ٤ [الاربعة 307 || A || ٢٤ [اربعة وعشرون 307 || A. add.
 اثنان 308 || A. add. من الزوج [وهو 308 || A ما يقابلها om., hic hab. seq. وبازائها 308 || D
 (sic) add. in ٢٨, ٢٨ in textu, ٨ [الثمانية والعشرة 310 || A || ٢٠ [عشرون 309 || A || ٢٢ [وعشرون
 [واحد 311 || A || ١٢ و [الاثنى عشر 310 || D. add. in marg. [في الجهة ... عشر 310 || A. marg.
 AD. عدد

(24) A: 15^v, 8-16^r, 6 (ed. ll. 700-716); D: 83^v, 3-83^v, 11.

[النوع 313 || A الخمسة الى ٩ والى ١٣ و ١٧] الخمسة الى التسعة والثلاثة عشر والسبعة عشر 312
 [يليه قبله 316 || D (homoeotel.) om. [واثبت ... يقابله 316-7 || A وما [ما 315 || D الجنس
 318 || D [انه نصفه 8] نصفه [وجدت 318 || A في الطرف [الطرف 317 || A نقابله

on the top; you put 3 in the left-hand middle cell and 23 opposite to it on the right.⁸⁷ Next, you put 2 in the upper right-hand corner and opposite to it diagonally, in the lower left-hand corner, its complement, namely 24. You put 4 in the upper left-hand corner and opposite to it diagonally, in the lower right-hand corner, its complement, namely 22. You put 6 on the right-hand side and its complement, 20, opposite to it on the left. You put 8 and 10 on the bottom, 12 on the right, and you put opposite to each its complement.

4	18	25	16	2
20	11	9	19	6
3	21	13	5	23
14	7	17	15	12
24	8	1	10	22

(Squares of orders $n = 4t + 1$, $t > 1$)

You pass now from five to nine, thirteen, seventeen and those of this kind.

(§24. Filling all first borders)

$\frac{n^2 - 1}{2}$	$n^2 + 1 - \Delta_1^C$		2	$\frac{n^2 - 5}{2}$
$n^2 + 1 - 4$				4
Δ_1^L				$n^2 + 1 - \Delta_1^L$
$\frac{n^2 + 7}{2}$	Δ_1^C		$n^2 + 1 - 2$	$\frac{n^2 + 3}{2}$

Put the last small even term in the upper left-hand corner of the first border — that is, the (border) following the inner square which you have (completely) filled with odd numbers — and opposite to it diagonally, in the lower right-hand corner of the first border, its complement. Put the preceding small term in the upper right-hand corner of the first border, and opposite to it diagonally, in the lower left-hand corner, the term which is its complement.⁸⁸ [Once you have done that, you will find that the

⁸⁷ As taught in §9.

⁸⁸ With this, the former excesses Δ_1^L and Δ_1^C (§14) are changed, according to the equalization rule seen in §22, into $\Delta_1^{L*} = \Delta_1^L - 4$, $\Delta_1^{C*} = \Delta_1^C - 2$ (see table).

فعلت ذلك وجدت زيادة الصف الاعلى ستة عشر وزيادة الصف الايمن
ستة عشر فصارت زيادة الصف الاعلى مثل زيادة الصف الايمن.]

فأثبت الاثنين في الصف الاعلى وانظر الى كم زيادة الصف الاعلى
فخذ نصفها ثم عدّ بعد الاثنين مثل ذلك النصف من الأزواج القليلة
حيث انتهيت فأثبت العدد الذى تنتهى اليه فى الصف الاسفل وأثبت
بازاء كلّ واحد منهما الطرف الذى يقابله. وأثبت الاربعة فى الجهة اليمنى
ثم عدّ بعد الاربعة مثل نصف الزيادة (من الأزواج القليلة) فحيث
انتهيت فأثبت العدد الذى تنتهى اليه فى الجهة اليسرى وأثبت بازاء كلّ
واحد منهما الطرف الذى يقابله.
فاذا فعلت ذلك فقد عدلت الصف الاول من جميع الجهات فى هذا
النوع من المربعات.

(25) ثم أثبت الطرفين القليلين اللذين يكون نقصانهما عن حقيهما اذا
جُمعا اثنى عشر فى الزاويتين اليمنتين من الصف الثانى وليكن اقلهما فى
الزاوية العليا وأثبت بازاءهما قطراً فى الزاويتين اليسرائين ما يقابلهما.

الاعلى اربعة A, (et ٤ hab. in marg.) على الاول corr. ex) الاعلى ٤ [الاعلى ستة عشر
الى 320 A || ٢ الاثنين 320 AD || وصارت [فصارت 319 D || اثنين A, ٢ [ستة عشر 319 D ||
الاربعة 324 A || ٤ [الاربعة 323 AD || منها [منهما 323 A || الصف [النصف 321 D || om.
واست [وأثبت 325 A || من الصف الاسفل [فى الجهة اليسرى 325 D || الزيادة A, ذلك اعنى ٤
cum erroribus) وأثبت ٤ (...) الطرف الذى يقابله deinde iter. verba [الذى يقابله 326 D ||
الجنس [النوع 328 A || عادت [عدلت 327 A || (ذلك اعنى sed sine D.

(25) A: 16^f, 6-16^f, 13 (ed. ll. 717-724); D: 83^v, 11-83^v, 16.

331 D || المسمن A, الممتين [اليمنتين 330 A || ١٢ [اثنى عشر 330 D || است [أثبت 329
[اليسرائين 331 AD || ما يقابلها post [قطراً 331 A || بازائها 331 A || فأثبت [وأثبت

excess of the upper row is 16, the excess of the right-hand row 16, thus the excess of the upper row equals the excess of the right-hand row.]⁸⁹

	Δ_1^L	Δ_1^C	$\frac{n^2-1}{2}$	$\frac{n^2+3}{2}$	$\frac{n^2-5}{2}$	Δ_1^{L*}	Δ_1^{C*}
$n = 9$	20	18	40	42	38	16	16
$n = 13$	28	26	84	86	82	24	24
$n = 17$	36	34	144	146	142	32	32

Put then 2 in the upper row; consider the (new) excess of the upper row, take its half, and count after 2 as many small even (numbers) as this half, and, having arrived there, put the number reached in the lower row; put opposite to each of these two numbers its complementary term. Put 4 on the right side; then count after 4 as many small even numbers as half of the excess, and, having arrived there, put the number reached in the left-hand side; put opposite to each of these two numbers its complementary term.⁹⁰

	Δ_1^{L*}	Δ_1^{C*}	$2 - \Delta_1^C$	$4 - \Delta_1^L$
$n = 9$	16	16	-16	-16
$n = 13$	24	24	-24	-24
$n = 17$	32	32	-32	-32

Once you have done that, you will have equalized all the sides of the first border in this kind of square.

(§25. First steps for filling all second borders)

(i) Put then the pair of small terms the sum of which is less than their sum due by 12 in the right-hand corners of the second border, with the lesser one above; put in the diagonally opposite corners, on the left, their complements.⁹¹

⁸⁹ Early reader's gloss illustrating the situation for $n = 9$ after placing 40, 42, 38. As a matter of fact, this equality is verified generally for the new excesses (the terms thus placed cancel out the difference $\Delta_1^L - \Delta_1^C = 2$).

⁹⁰ First, it comes to the same to count $\frac{1}{2} \cdot \Delta_1^{L*}$ even numbers after 2 or Δ_1^{L*} (natural) numbers after 2, and the same holds for counting $\frac{1}{2} \cdot \Delta_1^{C*}$ and Δ_1^{C*} after 4. Second, the two numbers thus reached will be equal to Δ_1^C in the first case and Δ_1^L in the second; it could therefore have been simply said that we are to put, in the rows opposite to those of 2 and 4, quantities equal to the initial excesses Δ_1^C and Δ_1^L , respectively, the placement of the complements thus reducing the excesses by $|2 - \Delta_1^C|$ and $|4 - \Delta_1^L|$. By the way, using the *initial* excesses and deficits would be in keeping with their use in the subsequent paragraphs. It is therefore likely that the original text was corrupt at this place, with later attempts to restore it.

⁹¹ See the figure below, showing the placing of the numbers mentioned in §§25–27. The subsequent table gives, for the order considered, the sum due for two cells, then this sum due less 12, as required, from which we find the pair of consecutive small numbers α_2 , $\alpha_2 + 2$ attributed to the right-hand corners of the second border.

ثمّ انظر العددين القليلين اللذين اذا جُمعا قصّرا عن حقّهما باقلّ من
زيادة الصفّ الثانى الايمن باربعة عشر فاثبتهما فى الصفّ الثانى من الجهة
اليمنى وبازاءهما فى اليسرى ما يقابلهما.

335 ثمّ اطلب العددين القليلين اللذين اذا جُمعا قصّرا عن حقّهما باقلّ من
زيادة الصفّ (الثانى) الاعلى بثمانية فاثبتهما فى ذلك الصفّ وبازاءهما فى
الاسفل ما يقابلهما.

كان الصواب انه: *om. D, hic hab.:* [باربعة عشر ... اليمنى 333-4 || *D* اليسريين, *A* اليسرايين
باربعة 333 || باثنى عشر واربعه عشر وهما موجودان ابدا فاثبتهما فى الجهة اليمنى فى الصف الثانى
|| *add. in marg. D* || القليلين 335 || *om. A* || ما يقابلهما 334 || *A* فى [من 333 || *A* ١٤] عشر
D. الصف الاعلى [ذلك الصفّ 336 || *A* ٨] بثمانية 336

$n^2 + 1 - (\alpha_i + 2)$	$\gamma_i + 2$	γ_i	α_i
$n^2 + 1 - \beta_i$			β_i
$n^2 + 1 - (\beta_i + 2)$			$\beta_i + 2$
$n^2 + 1 - \alpha_i$	$n^2 + 1 - (\gamma_i + 2)$	$n^2 + 1 - \gamma_i$	$\alpha_i + 2$

	$n^2 + 1$	$(n^2 + 1) - 12$	$\alpha_2, \alpha_2 + 2$
$n = 9$	82	70	34, 36
$n = 13$	170	158	78, 80
$n = 17$	290	278	138, 140

(ii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the second right-hand row less 14; put them in the second border on the right and, opposite to them on the left, their complements.⁹²

	$n^2 + 1$	Δ_2^C	$(n^2 + 1) - (\Delta_2^C - 14)$	$\beta_2, \beta_2 + 2$
$n = 9$	82	34	62	30, 32
$n = 13$	170	50	134	66, 68
$n = 17$	290	66	238	118, 120

46				77		28	26	34
52	40	64	71	69	3	2	38	30
50	78	27	31	61	63	23	4	32
	9	29	39	37	47	53	73	
7	15	25	49	41	33	57	67	75
	81	65	35	45	43	17	1	
	20	59	51	21	19	55	62	
	44	18	11	13	79	80	42	
48				5		54	56	36

$n = 9$

(iii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the second upper

⁹² For the values of Δ_2^C and Δ_2^L in the tables to (ii) and (iii), see § 14. Same for §§ 26–27.

(26) ثمّ اثبت العددين القليلين اللذين يكون نقصانهما عن حقّهما اذا
 جُمعا عشرين في الزاويتين اليمنتين من الصفّ الثالث وليكن اقلّهما في
 العليا واثبت بازاءهما قطراً في الزاويتين من الصفّ الثالث يسراً ما يقابلهما.
 340 ثمّ اطلب العددين القليلين اللذين اذا جُمعا قصراً عن حقّهما باقلّ من
 زيادة الصفّ الثالث الايمن باثنين وعشرين فاثبتهما في هذا الصفّ
 وبازاءهما في الصفّ الثالث الايسر ما يقابلهما.
 ثمّ اطلب العددين القليلين اللذين اذا جُمعا قصراً عن حقّهما باقلّ من
 345 زيادة الصفّ الثالث الاعلى بثمانية فاثبتهما في هذا الصفّ وبازاءهما في
 الصفّ الثالث <الاسفل> ما يقابلهما.

(26) A: 16^r, 13-16^r, 21 (ed. ll. 725-732); D: 83^v, 17-83^v, 22.

A, اليمنتين [اليمنتين 339 || A ٢٠] عشرين 339 || A, يكونان [يكون 338 || D است] اثبت 338
 A || هل [باقلّ 344 || A om.] القليلين 344 || A ٢٢ [باثنين وعشرين 342 || D اليمين
 345 فاست هما, A, فسهما] فاثبتهما 345 || A ٨ [بثمانية 345 || D الايمن الاعلى] الاعلى 345

row less 8; put them in this row and, opposite to them below, their complements.

	$n^2 + 1$	Δ_2^L	$(n^2 + 1) - (\Delta_2^L - 8)$	$\gamma_2, \gamma_2 + 2$
$n = 9$	82	36	54	26, 28
$n = 13$	170	52	126	62, 64
$n = 17$	290	68	230	114, 116

(§ 26. First steps for filling all third borders)

(i) Put then the pair of small numbers which have a sum less than their sum due by 20 in the right-hand corners of the third border, with the lesser one above; put, opposite to them diagonally, in the corners of the third border on the left, their complements.

	$n^2 + 1$	$(n^2 + 1) - 20$	$\alpha_3, \alpha_3 + 2$
$n = 13$	170	150	74, 76
$n = 17$	290	270	134, 136

(ii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the third right-hand row less 22; put them in this row and, opposite to them in the third left-hand row, their complements.

	$n^2 + 1$	Δ_3^C	$(n^2 + 1) - (\Delta_3^C - 22)$	$\beta_3, \beta_3 + 2$
$n = 13$	170	74	118	58, 60
$n = 17$	290	98	214	106, 108

94							161						52	50	74
112	90						155	145	7			64	62	78	58
110	104	84	144	151	139	137	23	3	2			82	66	60	
	102	166	51	59	55	125	127	131	47	4		68			
		17	57	71	75	105	107	67	113	153					
	13	29	53	73	83	81	91	97	117	141	157				
11	27	35	49	69	93	85	77	101	121	135	143	159			
		165	149	129	109	79	89	87	61	41	21	5			
		169	133	103	95	65	63	99	37	1					
		28	123	111	115	45	43	39	119	142					
		88	26	19	31	33	147	167	168	86					
	92				15	25	163		106	108	80				
96							9						118	120	76

$n = 13$

(iii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the third upper row

(27) ثمّ اثبت العددين القليلين اللذين يكون نقصانهما عن حقّهما اذا
 جُمعا ثمانية وعشرين في الزاويتين اليمينتين من الصفّ الرابع وليكن اقلّهما
 في العليا وبازاءهما قطراً في اليسرى ما يقابلهما.
 ثمّ اطلب العددين القليلين اللذين اذا جُمعا قصراً عن حقّهما باقلّ من
 زيادة الصفّ الرابع الايمن بثلاثين فاثبتهما في هذا الصفّ وبازاءهما في
 الصفّ الرابع الايسر ما يقابلهما.

(27) A: 16^f, 21 - 16^v, 7 (ed. ll. 733-742); D: 83^v, 22 - 84^f, 2.

[اليمنتين 348 || A ٢٨] ثمانية وعشرين 348 || D || (sic) الذين [الذين 347
 (اثنين sc.) ٢] بثلاثين 351 || D قصراً [قصراً 350 || D است] اطلب 350 || D المنين A, اليمائين
 ثمّ 356 || A ٨ [ثمانية 354 || D الاعلى الرابع] الرابع الاعلى 354 || D om. [الصفّ 352 || A

	$n^2 + 1$	Δ_3^L	$(n^2 + 1) - (\Delta_3^L - 8)$	$\gamma_3, \gamma_3 + 2$
$n = 13$	170	76	102	50, 52
$n = 17$	290	100	198	98, 100

(i) Put then the pair of small numbers which have a sum less than their sum due by 28 in the right-hand corners of the fourth border, with the lesser one above, and opposite to them diagonally on the left their complements.

	$n^2 + 1$	$(n^2 + 1) - 28$	$\alpha_4, \alpha_4 + 2$
$n = 17$	290	262	130, 132

[illegible]
$$n = 17$$

(ii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the fourth right-hand row less 30; put them in this row and, opposite to them in the fourth left-hand row, their complements.

ثمّ اطلب العددين القليلين اللذين اذا جُمعا قصّرا عن حقّهما باقلّ من
زيادة الصفّ الرابع الاعلى بثمانية فاثبتهما فى ذلك الصفّ وبازاءهما فى
الاسفل ما يقابلهما.

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ثمّ افعل كذلك ابداً زيادةً ثمانية ثمانية فى الصفّ الايمن وفى الاعلى
على حاله بنقصان ثمانية ابداً.

(28) فاذا فعلت ذلك فقد صارت زيادة كلّ صفّ من الصفوف العليا
على حقّه ستّة وزيادة كلّ صفّ من اليمنى على حقّه اثنين ويبقى من
البيوت الفارغة فى كلّ صفّ من الصفوف الثانية اربعة ابيات وفى كلّ
صفّ من الثالثة ثمانية ابيات وفى كلّ صفّ من الرابعة اثنا عشر وكذلك
ابداً زيادةً اربعة اربعة.

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(29) فتعمد حينئذٍ الى ما يبقى من الاعداد فتأخذ كلّ اربعة منسّقة من
القليلة فتثبت الاول منها فى صفّ من الصفوف العليا والثالث فى الصفّ
الذى يقابله من السفلى والثانى فى اليمنى والرابع فى اليسرى وتثبت بازاء
كلّ واحد من هذه الاربعة ما يقابله حتّى ترسم فى كلّ صفّ من
الصفوف مثل ذلك.

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357 || A حياله [على حاله 357 || A ٨ ٨] ثمانية ثمانية 356 || A وكذلك فافعل [افعل كذلك
A. ايضاً] ابداً 357 || A ٨ ٨] ثمانية

(28) A: 16^v, 7-16^v, 11 (ed. ll. 743-747); D: 84^r, 2-84^r, 5.

ثمانية 361 || A (sic) ٨ [اربعة 360 || A ٢] اثنين 359 || A ٦ [ستّة 359 || D صار] صارت 358
[وكذلك 361 || D اثني عشر, A (sic) ١٦] اثنا عشر 361 || D om. [فى كلّ صفّ 361 || A ٨
A. ٤ ٤] [اربعة اربعة 362 || A وهكذا

(29) A: 16^v, 12-16^v, 16 (ed. ll. 748-752); D: 84^r, 5-84^r, 7.

من الصف [فى صفّ من 364 || D العلة] [القليلة 364 || A om. [منسّقة 363 || A ٤] [اربعة 363
A. البانى] [الذى 365 || D om. صفّ من, A. من

	$n^2 + 1$	Δ_4^C	$(n^2 + 1) - (\Delta_4^C - 30)$	$\beta_4, \beta_4 + 2$
$n = 17$	290	130	190	94, 96

(iii) Then look for the pair of small numbers which have a sum less than their sum due by (an amount equal to) the excess of the fourth upper row less 8; put them in this row and, opposite to them below, their complements.

	$n^2 + 1$	Δ_4^L	$(n^2 + 1) - (\Delta_4^L - 8)$	$\gamma_4, \gamma_4 + 2$
$n = 17$	290	132	166	82, 84

Proceed always likewise: for the right-hand row(s), by successive additions of 8, and, as for the upper one(s), always with a uniform subtraction of 8.⁹³

(§ 28. Remaining excesses, deficits, and empty cells)

Once you have done that, the excess of each upper row over its sum due will be 6, the excess of each right-hand one over its sum due will be 2, and the number of remaining empty cells will be four in each second row, eight in each third row, twelve in each fourth row, and so on always by adding successively four.

	Δ_i^{L**}	Δ_i^{C**}	N_2^{**}	N_3^{**}	N_4^{**}
$n = 9$	6	2	4	8	12
$n = 13$	6	2	4	8	12
$n = 17$	6	2	4	8	12

(§ 29. Reducing these differences to ± 2)

	$n^2 + 1 - (\delta + 4)$	δ		$\Rightarrow n^2 + 1 - 4$
$n^2 + 1 - (\delta + 2)$			$\delta + 2$	
$\delta + 6$			$n^2 + 1 - (\delta + 6)$	
	$\delta + 4$	$n^2 + 1 - \delta$		$\Rightarrow n^2 + 1 + 4$
$\Downarrow$			$\Downarrow$	
$n^2 + 1 + 4$			$n^2 + 1 - 4$	

You will now turn your attention to the remaining numbers. You take groups of four consecutive small (even numbers) and put the first one in

⁹³ For determining the β_i , the quantity to be subtracted from the excesses increases by 8, whereas, for determining the γ_i , this quantity remains invariably equal to 8.

(30) فاذا فعلت ذلك نظرت الى عددين من القليلة فاثبت الاول منهما في الجهة العليا في صف من الصفوف والثاني في الاسفل وبازاءهما ما يقابلهما. ثم اعمد الى عددين آخرين فاثبت الاول منهما في الجهة اليسرى من ذلك الصف والثاني في اليمنى وبازاءهما ما يقابلهما. وافعل مثل ذلك بجميع الصفوف.

فاذا فعلت ذلك فقد صار كل صف معادلاً للذي يقابله لا يزيد بعضها على بعض شيئاً.

(31) فان بقي من الايات شيء فارغ فائما يبقى اربعة ايات تقابل اربعة او ثمانية تقابل ثمانية او اثنا عشر تقابل اثني عشر <وكذلك ابداً بزيادة اربعة اربعة> فتعدل كل اربعة منها باربعة اعداد منسقة من الباقية على المثال الذي شرحنا [في اول الباب] او باربعة اعداد يكون كل عددين منها منسقين فتثبت اول الاولين وثاني الآخرين في جهة وثاني الاولين واول الآخرين في الجهة التي تقابلها وبازاء كل واحد منها العدد الذي يقابله.

(30) A: 16^v, 16-16^v, 21 (ed. ll. 753-759); D: 84^r, 8-84^r, 11.

370 A || om. [في صف ... يقابلهما 369-70 A || اليسرى [العليا 369 D || فاست] فاثبت 368 D. فاست [فاثبت

(31) A: 16^v, 21-17^r, 4 (ed. ll. 760-765); D: 84^r, 11-84^r, 14.

اثنا 376 A || ٤ او ٨ تقابل ١٨ و [اربعة او ثمانية تقابل ثمانية او 376 D || om. [ايات 375 D فعل لكل A, تعدل كل [فتعدل كل 377 A || ١٢ [اثني عشر 376 D || ابني عشر A, ١٢ [عشر 378 D || hic et infra] متسقة [منسقة 377 A || om. [اعداد 377 A || ٤ (pr.) اربعة 377 || D. عدد [واحد 380 A || om. [او 378 D (v. enim D, fol. 33^v) || الكتاب [الباب

any upper row, the third in the opposite row below, the second on the right and the fourth on the left, and you put opposite to each of these four its complement.⁹⁴

(You proceed likewise) until you have done this for each border.

	Δ_i^{L***}	Δ_i^{C***}	N_2^{***}	N_3^{***}	N_4^{***}
$n = 9$	2	-2	2	6	10
$n = 13$	2	-2	2	6	10
$n = 17$	2	-2	2	6	10

(§ 30. Eliminating these last differences)

Once you have done that, you consider a pair of small numbers; put the first in any upper row, the second below (in the same border), and opposite to them their complements. Then turn your attention to another pair; put the first on the left side of the same border,⁹⁵ the second on the right, and opposite to them their complements. Do the same for all borders.⁹⁶

	$n^2 + 1 - (\epsilon + 2)$	ϵ		$\Rightarrow n^2 + 1 - 2$
	$\epsilon + 2$	$n^2 + 1 - \epsilon$		$\Rightarrow n^2 + 1 + 2$

Once you have done that, each row and its opposite will be equalized and none will exceed the other in any way.

	Δ_i^{L****}	Δ_i^{C****}	N_2^{****}	N_3^{****}	N_4^{****}
$n = 9$	0	0	0	4	8
$n = 13$	0	0	0	4	8
$n = 17$	0	0	0	4	8

(§ 31. Completing all the rows)

If there are remaining empty cells, it can (only) be four facing four, eight facing eight, twelve facing twelve (and so on always by adding successively four). You will then equalize them by groups of four, (either) using tetrads of still available consecutive numbers in the way already explained [at the beginning of the section]⁹⁷, or using tetrads of numbers consisting in pairs of consecutive ones, putting the first of the first pair

⁹⁴ Equalization rule of § 21, with $s = 2$.

⁹⁵ The left side is now in excess.

⁹⁶ Applying once again the equalization rule of § 21, this time with $s = 1$.

⁹⁷ § 20, at the beginning of the section on equalization rules by means of even numbers.

(32) والعمل في السبعة والاحد عشر والخمسة عشر والتسعة عشر وما
اشبه ذلك هو ان تثبت اخير الاطراف القليلة من الازواج في الزاوية العليا
اليسرى من الصفّ الاول المحيط بالمرّبع الداخل الذي حشوته بالافراد
وبازائه قطرًا في الزاوية اليمنى السفلى من هذا الصفّ ما يقابله. واثبت
385 الطرف القليل الذي قبله في الزاوية اليمنى العليا من هذا الصفّ وبازائه
قطرًا في الزاوية اليسرى السفلى ما يقابله.
فاذا فعلت ذلك فقد عدلت الصفّ الاول في جميع المربعات من هذا
النوع.

(33) ثمّ انظر [بعد ذلك] كم زيادة كلّ صفّ من الصفوف العليا واطلب
390 عددين من القليلة اذا جمعا نقصا مثلها فاثبتهما في زاويتي ذلك الصفّ
وليكن اقلّهما في اليمنى وبازائهما قطرًا في السفلى ما يقابلهما حتّى تأتي
على جميع زوايا الصفوف الباقية.

(32) A: 17^r, 4-17^r, 11 (ed. ll. 766-773); D: 84^r, 15-84^r, 19.

tit. D : الفصل الخامس

D || آخر A, اخر [اخير 383 || وما شاكلها] وما اشبه ذلك 3-382 || om. [والتسعة عشر 382
|| ما 387 || A. om. [السفلى 387 || D وست] واثبت 385 || AD add. الباقية [اليسرى 384
D. الجنس] النوع 389 || D في [من 388 || AD من] في 388 || D وادا [فاذا 388 || D الطرف الذي

(33) A: 17^r, 11-17^r, 17 (ed. ll. 774-780); D: 84^r, 19-84^r, 23.

A || اليمين [اليمنى 392 || D مثلها] مثلها 391 || A حمّا [جمعا 391 || A العليلين] القليلة 391
|| D ما يقابلها قطرًا في السفلى A, في السفلى ما يقابلها قطرًا [قطرًا في السفلى ما يقابلها 392
395 || A كلّ ما يبقى] ما بقى 394 || A. om. [الصفوف 393 || D add. in marg. على 393

and the second of the second pair on one side, the second of the first pair and the first of the second pair on the facing side, and opposite to each its complement.

(Squares of orders $n = 4t + 3$)⁹⁸

Treatment for (the squares of) seven, eleven, fifteen, nineteen and the like.⁹⁹

(§ 32. Completing all first borders)

You put the last of the small even terms in the upper left-hand corner of the first border, which surrounds the inner square which you have filled with odd numbers, and opposite to it diagonally, in the lower right-hand corner of the same border, its complement. Put the preceding small term in the upper right-hand corner of the same border and, opposite to it diagonally, in the lower left-hand corner, its complement.¹⁰⁰

	$\frac{n^2-1}{2}$	$\frac{n^2-5}{2}$	$\frac{n^2+1}{2}$	$\frac{n^2+3}{2}$	$\frac{n^2+7}{2}$
$n = 7$	24	22	25	26	28
$n = 11$	60	58	61	62	64
$n = 15$	112	110	113	114	116
$n = 19$	180	178	181	182	184

Once you have done that, you will have equalized the first border for all squares of this kind.

(§ 33. Equalizing the horizontal rows of all other borders)

Then consider [after that]¹⁰¹ the amount of the excess of each (remaining) upper row and look for the pair of small numbers such that their sum is less (than their sum due) by the same amount. Put them in the corner cells of this row, the lesser on the right, and opposite to them diagonally below their complements. You complete in this way all corners of the remaining borders.¹⁰²

⁹⁸ Subject of §§ 32–35.

⁹⁹ For the orders $n = 4t + 1$, the largest explicitly mentioned order was $n = 17$; here, it is $n = 19$, which is indeed the largest constructed (p. 87). By the way, we may observe that no square larger than 20×20 is represented in the treatise.

¹⁰⁰ Placing $\frac{n^2-1}{2}$ & $\frac{n^2-5}{2}$ (on the top), and $\frac{n^2+1}{2}$ & $\frac{n^2+3}{2}$ (on the right), then their complements, will eliminate the uniform differences $\Delta_1^L = 4$, $\Delta_1^C = 2$ (§ 17).

¹⁰¹ Either the gloss already seen (§§ 6 & 9) or to be inserted after ‘remaining empty cells’ below (as in § 34 — if so then not a gloss there).

¹⁰² The pair α_i , $\alpha_i + 2$ is put in the upper corners of the i th border ($i > 1$). As we are told, their sum $2\alpha_i + 2$ is less than their sum due $n^2 + 1$ by Δ_i^L (see table); thus $n^2 + 1 - \Delta_i^L = 2\alpha_i + 2$, and this determines their individual values. The upper horizontal rows will then no longer display any excess (or, for the opposite, deficit).

فاذا فعلت ذلك فقد عدلت الصفوف العليا والسفلى كلها وكان ما بقى
 395 من ابياتها الفارغة اربعة تقابل اربعة او «ثمانية تقابل ثمانية وكذلك»
 زيادة اربعة اربعة فتعدل كل اربعة منها باربعة اعداد على ما شرحنا فيما
 تقدم.

(34) ويبقى الصفوف اليمنى [تزيد على حقها اما ستة عشر واما اربعة
 وعشرين واما اثنين وثلاثين واما اربعين واما ثمانية واربعين وما يخرج من
 400 ثمانية ثمانية] ويبقى «من» الابيات الفارغة اربعة تقابل اربعة او ثمانية
 تقابل ثمانية وكذلك بزيادة اربعة اربعة. فتطلب عددين من الاعداد
 الكثيرة اذا جُمعا زادا على حقهما زيادة اذا اضيفت الى زيادة الصف
 الايمن كانت مساوية لنقصان عددين من القليلة اذا جُمعا. فاثبت العددين
 الكثيرين في الجهة اليمنى والعددين القليلين في هذه الجهة واثبت في
 405 اليسرى ما يقابل الاربعة الاعداد.

اربعة 396 || A و [او 395 || D (*homeotel.*) om. (تقابل اربعة A, 4 تقابل 4 [اربعة تقابل اربعة
 في صدر الكتاب [فيما تقدم 396-7 || A اربعة [باربعة 396 || A 4 [اربعة 396 || A 4 4 [اربعة
 D (*v. enim D, fol. 33^v*).

(34) A: 17^r, 17-17^v, 6 (ed. ll. 781-790); D: 84^r, 23-84^v, 5.

اربعة 398-9 || A ١٦ [ستة عشر 398 || D *add. in marg.* D || 398 على 398 || D زيد [تزيد 398
 399 || A ٣٢ [اثنين وثلاثين 399 || D او [واما 399 || A (*homoeotel.*) om. (وعشرين واما
 A ٨ ٨ [ثمانية ثمانية 400 || AD لا [وما 399 || A ٨ [ثمانية واربعين 399 || A (*sic*) 4 [اربعين
 اربعة تعادل اربعة A, 4 تعادل 4 و ٨ تعادل ٨ [اربعة تقابل اربعة او ثمانية تقابل ثمانية 400-1 ||
 زادت [زادا 402 || D الكثير [الكثيرة 402 || A 4 4 [اربعة اربعة 401 || D او ثمانية تعادل ثمانية
 403 || A العليا [القليلة 403 || D متساوية [مساوية 403 || A برياده [زيادة 402 || D زاد A,
 404 || A (*homoeotel.*) om. (الكثيرين في الجهة اليمنى والعددين 404 || D فاست [فاثبت
 كل عدد بازاء A. *add.* كل عددين بازاء ما يقابلهما [الاربعة الاعداد 405 || D الكثيرين [الكثيرين

	$(n^2 + 1) - \Delta_2^L$	$\alpha_2, \alpha_2 + 2$	$(n^2 + 1) - \Delta_3^L$	$\alpha_3, \alpha_3 + 2$	$(n^2 + 1) - \Delta_4^L$	$\alpha_4, \alpha_4 + 2$
$n = 7$	$50 - 20 = 30$	14, 16				
$n = 11$	$122 - 28 = 94$	46, 48	$122 - 52 = 70$	34, 36		
$n = 15$	$226 - 36 = 190$	94, 96	$226 - 68 = 158$	78, 80	$226 - 100 = 126$	62, 64
$n = 19$	$362 - 44 = 318$	158, 160	$362 - 84 = 278$	138, 140	$362 - 124 = 238$	118, 120

Once you have done that, you will have equalized all the upper and lower rows, and the remaining empty cells will be four facing four, eight facing eight, and so on by adding successively 4. You will then equalize each group of four by means of four numbers in the way we have explained previously.¹⁰³

(§ 34. Equalizing the vertical rows of all other borders)

$\alpha_i + 2$		α_i
$n^2 + 1 - \gamma_i$		γ_i
$n^2 + 1 - (\gamma_i - 2)$		$\gamma_i - 2$
$n^2 + 1 - \beta_i$		β_i
$n^2 + 1 - (\beta_i + 2)$		$\beta_i + 2$
$n^2 + 1 - \alpha_i$		$n^2 + 1 - (\alpha_i + 2)$

There remains (to deal with) the right-hand rows [(which) exceed their sum due by 16, 24, 32, 40, 48, and what results from successive (additions of) 8]¹⁰⁴ and there remain as empty cells four facing four, eight facing eight, and so on by successive additions of 4.¹⁰⁵ You then look for a pair of large numbers such that their sum exceeds their sum due by an amount which, when added to the excess of the right-hand row, equals the deficit of the sum of two small numbers. Put then the two large numbers on the right-hand side, and the two small numbers on this same side, and put on the left the complements of the four numbers.¹⁰⁶

¹⁰³ See §§ 20, 31.

¹⁰⁴ Gloss, perhaps by the same early reader as in § 19. For these values, see table below.

¹⁰⁵ Thus, in order to be left in the vertical rows with neutral placings, we shall have to fill four more cells (see figure), this being indeed applicable from the smallest order $n = 7$.

¹⁰⁶ Let, for the right-hand row of the i th border, Δ_i^{C*} be the new excess (after filling the corners), B_i the sum of the required pair of small numbers $\beta_i, \beta_i + 2$, and G_i the sum of the required pair of large numbers $\gamma_i, \gamma_i - 2$. We are told that $G_i - (n^2 + 1) + \Delta_i^{C*} = (n^2 + 1) - B_i$, that is, $B_i + G_i = 2(n^2 + 1) - \Delta_i^{C*}$ (which indeed eliminates the remaining excess), and therefore $\beta_i + \gamma_i = (n^2 + 1) - \frac{1}{2} \Delta_i^{C*}$. See the table, with the values set for β_i and γ_i .

فاذا فعلت ذلك فقد عدلت كل جهة من هاتين الجهتين بصاحبتهما وما
يبقى من الابيات [بعد ذلك] فارغاً فاربعة تقابل اربعة او ما زاد اربعة
اربعة فعدّلها كما وصفنا فيما تقدّم.

(35) ومثال العمل في الجهة اليمنى في مربع سبعة اثنك تجد زيادة الصف
الاعلى عشرين وزيادة الايمن ثمانية عشر فاذا اثبتت العددين اللذين
410 ينقصان عن حقهما بعشرين في الزاويتين العلياين وصيرت اقلهما في
اليمنى وبازاءهما ما يقابلهما قطراً عدلت العليا والسفلى وصارت زيادة
اليمنى ستة عشر فتنظر الى العددين الكثيرين اللذين تكون زيادتهما اثني
عشر [لاتك اذا اضفت الاثنى عشر الى الستة عشر التي هي زيادة اليمنى
415 صارت الزيادة ثمانية وعشرين معادلة لنقصان عددين من القليلة اذا

A, (وصحبه *corr. ex*) وصاحبتهما [بصاحبتهما 406 *D* || عادلّت [عدلت 406 *D* || *add.* ما يقابله
فاربعة 407 *A* || فارعه [فارغاً 407 *D* || الاعداد [الابيات 407 *D* || بقى [يبقى 407 *D* || صاحبهما
بعدها [فعدّلها 408 *A* || ٤ ٤ [اربعة اربعة 8-407 *A* || و [او 407 *A* || ٤ ٤ [تقابل اربعة
٤ *D* (v. enim *D*, fol. 33^v). في صدر الكتاب [فيما تقدّم 408 *A* ||

(35) *A*: 17^v, 6-17^v, 16 (ed. ll. 791-804); *D*: 84^v, 5-84^v, 12.

A || واذا [فاذا 410 *A* || (sic) ٢٨ [ثمانية عشر 410 *A* || ٢٠ [عشرين 410 *A* || ٧ [سبعة 409
A, العليايتين [العلياين 411 *D* || عشرين, *A* ٢٠ [بعشرين 411 *D* || است, *A* ايت [اثبتت 410
١٦ [ستة عشر 413 *D* || عدلت [عدلت 412 *A* || om. [ما يقابلهما ... اليمنى 3-412 *D* || العلين
A || ١٢ [الاثنى عشر 414 *A* || ١٢ [اثنى عشر 4-413 *D* || الكبيرين [الكثيرين 413 *D* || عشر, *A*
D القله [الليلة 415 *A* || (sic) ٢٦ [ثمانية وعشرين 415 *A* || ١٢ (corr. ex ١٢) [الستة عشر 414

	$\Delta_i^{C^*}$	$n^2 + 1$	$\frac{1}{2}\Delta_i^{C^*}$	$\beta_i + \gamma_i$	$\gamma_i - 2, \gamma_i$	$\beta_i, \beta_i + 2$
$n = 7$	16	50	8	42	30, 32	10, 12
$n = 11$	24	122	12	110	66, 68	42, 44
	48	122	24	98	70, 72	26, 28
$n = 15$	32	226	16	210	118, 120	90, 92
	64	226	32	194	122, 124	70, 72
	96	226	48	178	126, 128	50, 52
$n = 19$	40	362	20	342	186, 188	154, 156
	80	362	40	322	190, 192	130, 132
	120	362	60	302	194, 196	106, 108
	160	362	80	282	198, 200	82, 84

Once you have done that, you will have equalized each side of this pair of sides with its conjugate, and the remaining empty cells [after that] will be four facing four, or further multiples of 4. Equalize them as we have explained previously.

(§ 35. Example)

16				45			14
18	24	39	37	3	22	32	
20	9	23	21	31	41	30	
7	15	33	25	17	35	43	
40	49	19	29	27	1	10	
38	28	11	13	47	26	12	
36				5			34

$n = 7$

Example of the treatment for the right-hand side of the square of seven. You find that the excess of the (second) upper row is 20 and the excess of the right-hand one, 18.¹⁰⁷ Putting the two numbers (which have a sum) less than their sum due by 20 in the two upper corners,¹⁰⁸ with the lesser one on the right, and, opposite to them diagonally, their complements, you will have equalized the top and bottom (rows) and the excess on the right-hand side will be 16.¹⁰⁹ You consider then the two large numbers with an excess of 12. [For if you add 12 to 16, which is the right-hand excess, this gives an excess of 28, equal to the deficit of

¹⁰⁷ These values are in fact known from § 17.

¹⁰⁸ According to § 33.

¹⁰⁹ As noted by the interpolator in § 34.

جُمعاً]. فتثبت هذين العددين في الجهة اليمنى وتثبت في هذه الجهة أيضاً العددين القليلين اللذين اذا جُمعاً قصراً عن حقهما بثمانية وعشرين وتثبت في اليسرى بازاء كلّ واحد منها ما يقابله. وتعمل مثل ذلك في سائرهما.

وقد رسمتُ المربعات على هذه الشريطة منسوقة من الثلاثة الى تسعة 420 عشر لتقف عليها وتستدلّ بها على ما بعد ذلك.

يد	ب	مو	مه	مد	ح	يو
لب	كب	ج	لز	لط	كد	بج
ل	ما	لا	كا	كج	ط	ك
مج	له	يز	كه	لج	يه	ز
ى	ا	كز	كط	بط	مط	م
يب	كو	مز	بج	يا	كج	لح
لد	مح	د	ه	و	مب	لو

ب	يو	كه	بج	د
و	بط	ط	يا	ك
كج	ه	بج	كا	ج
يب	يه	يز	ز	يد
كب	ى	ا	ح	كد

ب	ط	د
ز	ه	ج
و	ا	ح

419 || D || منها 418 || A || (sic) ١٨ ١٨ [بثمانية وعشرين 417 || A || قصراً [قصراً 417 || من الثلاثة 420-21 || D || ان شا الله [في سائرهما 419 || D || بلثل [مثل 419 || A || والعمل [وتعمل 421 || A || على الواحد وعلى ان افرادها في داخلها فيستدل [الى تسعة عشر لتقف عليها وتستدلّ (sic) انشا hab. A. على ما لم ترسم بعد فراغنا من ذكر الازواج بمعونة الله تعالى [على ما بعد ذلك add. D. الله

(3 × 3)

hic omm. codd.

(5 × 5) A : 23^r (ed. p. 309); D : 84^v.

tit. A : ٥ في مثلها افرادها في وسطها :

tit. D : ٦٥ لوح خمسة بيوتها ٢٥ جملة ٣٢٥ كلّ صفّ منه ٦٥

(7 × 7) A : 23^r (ed. p. 309); D : 84^v.

tit. A : ٧ فى ٧ ايضاً :

tit. D : ١٧٥ لوح سبعة بيوته ٤٩ جملة ١٢٢٥ كلّ صفّ منه ١٧٥

D. ما (6, 4) له

You will proceed likewise for all other (squares of this kind).

I have represented the squares obeying this condition successively from (the order) three to nineteen; you will thereby become acquainted with them and rely on them for the subsequent ones.¹¹¹

2	9	4
7	5	3
6	1	8

$$n = 3$$

2	16	25	18	4
6	19	9	11	20
23	5	13	21	3
12	15	17	7	14
22	10	1	8	24

$$n = 5$$

14	2	46	45	44	8	16
32	22	3	37	39	24	18
30	41	31	21	23	9	20
43	35	17	25	33	15	7
10	1	27	29	19	49	40
12	26	47	13	11	28	38
34	48	4	5	6	42	36

$$n = 7$$

¹¹⁰ Our previous $G_i - (n^2 + 1) + \Delta_i^{C^*} = (n^2 + 1) - B_i$. This sentence, which attempts to complete the (obviously lacunary) reasoning of § 34 (see below, p. 199), must be an interpolation.

¹¹¹ We add the square of order three, missing here (in fact, superfluous), and repeat that of order 5, already represented (§23). For all, left to right orientation, thus different from the orientation used above (§§9–11, 23, 25–27, 35); this will facilitate the comparison with the squares constructed in our Commentary. For the squares of order $n = 4t + 3$, the place of the two ‘larger numbers’ given in §34 (our $\gamma_i - 2$, γ_i) may vary, either in their succession (with $\gamma_i - 2$ above, see case $n = 11$ with $i = 2$) or within the border (see case $n = 15$; same for the pair β_i , $\beta_i + 2$). See also p. 27, n. 37.

لد	ب	فيح	قيو	ح	فيج	ى	قى	غ	يو	لو
عب	مو	يخ	قب	ز	صز	قر	ق	كد	مح	ن
ع	سو	نخ	ج	كج	قط	صا	لج	س	نو	نب
كو	سح	قه	مح	لغ	فا	نا	مز	يز	ند	صو
كح	قط	صح	عج	سز	نز	نط	مط	كط	يخ	صد
فيا	صه	فز	عز	نخ	سا	سط	مه	له	كز	يا
ل	ه	كا	لز	سج	سه	نه	فه	قا	قيز	صب
ص	مب	ا	عه	لط	ما	عا	عط	قكا	ف	لب
فد	مد	سب	قيط	صط	لج	لا	يط	سد	عح	لح
م	عد	قد	ك	قيه	كه	يه	كب	صح	عو	فب
فو	قك	د	و	قيد	ط	قيب	يب	يد	قو	لغ

لد	كو	كح	و	عز	عب	يد	سو	مو
ل	لح	ب	ح	سط	عا	سد	م	نب
لب	د	لج	نح	سا	لا	كر	عح	ن
ح	عج	نح	مز	لز	لط	كط	ط	عد
عه	سز	نز	لج	ما	مط	كه	يه	ز
ع	ا	يز	مح	مه	له	سه	فا	يب
س	سب	نه	بط	كا	نا	نط	ك	كب
كد	مب	ف	عط	يخ	يا	بخ	مد	نخ
لو	نو	ند	عو	ه	ي	سح	يو	مح

tit. A : ٩ في ٩

لوحة تسعة في تسعة آخر الأعداد فيه ٨١ جملة ٣٣٢١ وكل صف منه ٣٦٩ *tit. D*

$$\mathcal{D}. \text{ع} [(8, 2) \text{مب}$$

tit. A : 11, 12, 13

٢٣٨١ وكلّ صفّ من صفوفه ٦٢١
 لوح احد عشر منسوقة على الواحد زيادةً واحد واحد وآخر الاعداد فيها ١٢١ وحمتها : *tit. D*

A latere quadrati in $\mathcal{A}$: $\mathfrak{V}\mathfrak{V}\mathfrak{V}$ ($= n^2 + 1$).

$\mathcal{A}$ سج $(8, 8)$ مج $\mathcal{A}$ ما $(5, 8)$ نا $\mathcal{A}$ قط $(6, 9)$ فط $\mathcal{A}$ قد $(8, 10)$ قب $\mathcal{A}$ قع $(3, 11)$ قح
 $\mathcal{A}$ قيز $\mathcal{A}$ قا $(11, 6)$ قيا $\mathcal{A}$ نه $(4, 6)$ مه $\mathcal{A}$ سا $(7, 7)$ سز $\mathcal{A}$ فز $(6, 7)$ نز $\mathcal{A}$ فه $(9, 8)$ قه
 $\mathcal{A}$ ما $(4, 4)$ عط $\mathcal{A}$ فز $(8, 5)$ لز $\mathcal{A}$ س $(7, 5)$ سح $\mathcal{A}$ عط $(3, 5)$ قا $\mathcal{A}$ قط $(2, 5)$
 $\mathcal{A}$ كد $(6, 2)$ كه $\mathcal{A}$ قد $(11, 3)$ فد $\mathcal{A}$ يح $(6, 3)$ لح $\mathcal{A}$ فا $(6, 4)$

34	26	28	6	77	72	14	66	46
30	38	2	3	69	71	64	40	52
32	4	23	63	61	31	27	78	50
8	73	53	47	37	39	29	9	74
75	67	57	33	41	49	25	15	7
70	1	17	43	45	35	65	81	12
60	62	55	19	21	51	59	20	22
24	42	80	79	13	11	18	44	58
36	56	54	76	5	10	68	16	48

 $n = 9$

34	2	118	116	8	113	10	110	108	16	36
72	46	18	102	7	97	107	100	24	48	50
70	66	58	3	23	89	91	103	60	56	52
26	68	105	43	83	81	51	47	17	54	96
28	109	93	73	67	57	59	49	29	13	94
111	95	87	77	53	61	69	45	35	27	11
30	5	21	37	63	65	55	85	101	117	92
90	42	1	75	39	41	71	79	121	80	32
84	44	62	119	99	33	31	19	64	78	38
40	74	104	20	115	25	15	22	98	76	82
86	120	4	6	114	9	112	12	14	106	88

 $n = 11$

عد	ن	نب	يد	قنب	لد	قسا	قلد	مب	فكو	فكد	مح	صد
نح	عح	سب	سد	و	ز	قه	قنه	قس	كب	قمو	ص	قيب
س	سو	فب	ب	ج	كج	قلز	قلط	قنا	قهد	فد	قد	قي
يو	سح	د	مز	قلا	فكز	فكه	نه	نط	نا	قسو	قب	قند
قن	ح	فتح	قيج	سز	قز	قه	عه	عا	نز	يز	قشب	ك
قلب	فز	قما	قيز	صز	صا	فا	فج	عج	نج	كط	يج	لح
قنط	قمح	قله	قكا	قا	عز	فه	صبح	سط	مط	له	كر	يا
م	ه	كا	ما	سا	فز	فط	عط	قط	فكط	قمط	قسه	قل
ند	قنح	ا	لز	صط	سج	سه	صه	فج	قلج	قسط	يب	قيو
قيد	قم	قنب	قبط	لط	مج	مه	قيه	قيا	فكج	كج	ل	نو
ق	لب	فو	قسخ	قسر	قمز	لج	لا	بط	كو	فج	قلح	ع
عب	ف	فج	قو	قسد	قسخ	كه	يه	ي	قمح	كد	صب	صع
عو	فك	قيح	قنو	يح	قلو	ط	لو	فكح	مد	مو	فكب	صو

$(13 \times 13) \mathcal{D} : 85^r$.

لوح ثلاثة عشر وآخر الاعداد فيه ١٦٩ وجملتها ١٤٣٦٥ وكل صف من صفوفه ١١٠٥ *tit.*:

$\mathcal{D}$. قكج (5,1) || قكح (3,6) قبط

74	50	52	14	152	34	161	134	42	126	124	48	94
58	78	62	64	6	7	145	155	160	22	146	90	112
60	66	82	2	3	23	137	139	151	144	84	104	110
16	68	4	47	131	127	125	55	59	51	166	102	154
150	8	153	113	67	107	105	75	71	57	17	162	20
132	157	141	117	97	91	81	83	73	53	29	13	38
159	143	135	121	101	77	85	93	69	49	35	27	11
40	5	21	41	61	87	89	79	109	129	149	165	130
54	158	1	37	99	63	65	95	103	133	169	12	116
114	140	142	119	39	43	45	115	111	123	28	30	56
100	32	86	168	167	147	33	31	19	26	88	138	70
72	80	108	106	164	163	25	15	10	148	24	92	98
76	120	118	156	18	136	9	36	128	44	46	122	96

$$n = 13$$

سب	ب	ركب	رك	ح	ى	ريد	ريج	ريب	يو	يخ	رو	رد	كد	سد
سو	ع	كو	قصح	قصو	لب	يا	قفط	رز	لد	قص	قصح	م	ف	قس
قنح	ند	صد	مب	قنب	ز	له	قنج	رج	قف	مح	صو	قعب	يخ	
قنب	قع	كك	قى	ج	لا	نا	قسه	قمر	قسط	قصب	قو	نو	عد	
عو	قصح	قيح	را	عه	قنط	قنه	قنج	لخ	فز	عط	كه	قخ	نخ	قن
فب	س	ره	قفا	قما	صه	قله	قلج	قخ	صط	فه	مه	كا	قسو	قمد
قنب	رط	قنه	قسط	قنه	ككه	قيط	قط	قيا	قا	فا	نزا	ما	يز	فد
ريا	قفز	قعا	قصح	قسط	قكط	قه	قيج	فكا	صز	عز	يخ	نه	لطا	يه
قم	ط	لج	مط	سط	فط	قيه	قيز	قز	قلز	قز	قعر	قصح	ريز	فو
لخ	فكد	ه	كط	سه	فكز	صا	صج	فكج	قلا	قسا	قصر	ركا	قب	قلح
فكح	فكب	ص	ا	قمر	سز	عا	عج	قحج	قلاط	قنا	ركه	قلو	قد	صح
فكو	ع	صب	قيد	كج	قصه	قعه	سا	نط	مز	كز	قيو	قلد	قنو	ق
ن	عب	قل	ققد	مد	ربط	قصا	نخ	مج	كج	مو	قصح	قلب	قند	قعو
نب	قمو	ر	كح	ل	قصد	ريه	لز	يط	قصب	لو	لح	قفو	قحج	قعد
قصب	ركد	د	و	ريخ	ريو	يب	يخ	يد	رى	رح	ك	كب	رب	قسد

(15 × 15) D : 85^v.

لوح خمسة عشر في خمسة عشر آخر الاعداد فيه ٢٢٥ وحملتها ٢٥٤٢٥ وكل صف من *tit.* : صفوفه ١٦٩٥

قلب || ١٣٤ *mut. ut vid. in* ١٣٣ (7,6) فكج || (v. *loculum infra*) فد (1,10) *corr. ex* قمد
 قح (2,2) قح || (v. *loculum supra*) قلد (3,3)

62	2	222	220	8	10	214	213	212	16	18	206	204	24	64
66	78	26	198	196	32	11	189	207	34	190	188	40	80	160
158	54	94	42	182	7	35	173	183	203	180	48	96	172	68
152	170	120	110	3	31	51	165	167	179	199	112	106	56	74
76	168	118	201	75	159	155	153	83	87	79	25	108	58	150
82	60	205	181	141	95	135	133	103	99	85	45	21	166	144
142	209	185	169	145	125	119	109	111	101	81	57	41	17	84
211	187	171	163	149	129	105	113	121	97	77	63	55	39	15
140	9	33	49	69	89	115	117	107	137	157	177	193	217	86
88	124	5	29	65	127	91	93	123	131	161	197	221	102	138
128	122	90	1	147	67	71	73	143	139	151	225	136	104	98
126	70	92	114	223	195	175	61	59	47	27	116	134	156	100
50	72	130	184	44	219	191	53	43	23	46	178	132	154	176
52	146	200	28	30	194	215	37	19	192	36	38	186	148	174
162	224	4	6	218	216	12	13	14	210	208	20	22	202	164

$$n = 15$$

قل	فب	فد	كب	رصد	مب	رمو	نح	رعز	رل	ركج	سد	سو	ركب	رك	عب	قنح
صد	قلد	صع	ق	يد	رعب	لح	يا	رنج	رعا	رن	عد	ريد	ريب	ف	قند	قصو
صو	قو	قلح	قيد	قيو	و	ز	له	رلز	رمز	رسز	رف	ل	رنج	قن	قند	قصد
كد	فح	قيح	قهب	ب	ج	لا	نا	ركط	رلا	رمج	ريج	رنو	قهد	قعب	قعب	رسو
رสบ	يو	قك	د	عط	ركج	ريط	ريه	ريج	فز	صا	صه	فج	رفو	قع	رعد	كح
رلو	رع	ح	رسه	قصز	قز	قصا	قفز	قفه	قيه	قيط	قيا	صع	كه	رفب	ك	ند
نو	رم	رسط	رهم	را	قعج	قكز	قسز	قسه	قله	قلا	قيز	فط	مه	كا	ن	رلد
فو	رعج	رمط	رلح	ره	قعز	قز	قنا	قما	قمج	قلج	قيج	فه	نز	ما	يز	رد
رعه	رنا	رله	ركز	رط	قفا	قسا	قلز	قمه	قنج	قكط	قط	فا	سج	نه	لط	يه
رب	ط	لح	مط	عج	قا	فكا	قمز	قمت	قلط	قسط	قفط	رما	ريز	رنا	رفا	فخ
ر	نب	ه	كط	سط	صز	قنط	قكج	فكه	قنه	قسج	قصج	ركا	رسا	رله	رلح	ص
صب	فكب	رعج	ا	سه	قمط	صط	قح	قه	قعه	قعا	قفج	ركه	رفط	يب	قسج	قصح
قب	قسو	رمد	رند	رز	سز	عا	عه	عز	رج	قصط	قصه	ريا	لو	مو	فكد	قفح
قفو	قسد	مح	قمو	فج	رفز	رلظ	سا	نط	مز	كز	لد	قمح	رمب	فكو	قد	
قف	قكح	قم	قعو	قعد	رفد	رفج	رنه	نج	مج	كج	ي	رس	لب	قنب	قنب	قي
قيب	قلو	قصب	قص	رعو	نج	رنب	رعب	لز	يط	م	ريو	عو	عح	ري	قنو	قنح
قلب	رح	رو	رنح	كو	رمح	مد	رلب	نج	س	سب	ركو	ركد	سج	ع	رنج	قس

(17 × 17) $\mathcal{D}$: 86^r.

لوح سبعة عشر في سبعة عشر آخر الاعداد فيه ٢٨٩ وجملتها ٤١٩٠٥ وكل صف من

صفوف منه ٢٤٦٥

١٣٦. [(8,8) قلط || ٢٦٩ (7,8) قسط || قعد ex corr.] (2,15) قفد .

130	82	84	22	264	42	246	58	277	230	228	64	66	222	220	72	158
94	134	98	100	14	272	38	11	253	271	250	74	214	212	80	154	196
96	106	138	114	116	6	7	35	237	247	267	280	30	258	150	184	194
24	108	118	142	2	3	31	51	229	231	243	263	256	144	172	182	266
262	16	120	4	79	223	219	215	213	87	91	95	83	286	170	274	28
236	270	8	265	197	107	191	187	185	115	119	111	93	25	282	20	54
56	240	269	245	201	173	127	167	165	135	131	117	89	45	21	50	234
86	273	249	233	205	177	157	151	141	143	133	113	85	57	41	17	204
275	251	235	227	209	181	161	137	145	153	129	109	81	63	55	39	15
202	9	33	49	73	101	121	147	149	139	169	189	217	241	257	281	88
200	52	5	29	69	97	159	123	125	155	163	193	221	261	285	238	90
92	122	278	1	65	179	99	103	105	175	171	183	225	289	12	168	198
102	166	244	254	207	67	71	75	77	203	199	195	211	36	46	124	188
186	164	48	146	288	287	259	239	61	59	47	27	34	148	242	126	104
180	128	140	176	174	284	283	255	53	43	23	10	260	32	152	162	110
112	136	192	190	276	18	252	279	37	19	40	216	76	78	210	156	178
132	208	206	268	26	248	44	232	13	60	62	226	224	68	70	218	160

$$n = 17$$

صع	ب	شنع	شو	ح	ی	شن	شمع	یو	شمه	یخ	شمب	شم	کد	کو	شلد	شلب	لب	ق
ر	قیح	لد	شکو	شکد	م	مب	شیح	یه	شیح	شطل	شیو	مح	ن	شی	ثع	نو	قک	قشب
نضع	قصو	قلح	نح	شب	ش	سد	یا	مز	رفط	ثز	شله	سو	رصد	رصب	عب	قم	قسو	قسد
فب	قصد	قصب	قنح	عد	رفو	ز	مج	عا	رعج	رفح	ثج	شلا	رغد	ف	قس	قع	قمح	رف
فد	قو	قص	قنح	قج	لط	سز	فز	رسه	رسز	رعط	رصلط	شکر	قف	قعد	قعب	رنو	رعح	
نکب	قح	قل	قفو	شکط	قیه	رنط	رنه	رنا	رمط	فکج	فکز	قلا	قیط	لج	قعو	رلب	رند	رم
	رلح	صد	قلب	شلج	شا	رلج	قمج	رکز	رکج	رکا	قنا	قنه	قمز	فکط	سا	کط	رل	رمح
	رلو	رسو	شلز	شه	رفا	رلز	رط	قسج	رج	را	قعا	قمز	قنح	فکه	فا	نز	که	صو
	فکح	ثما	شط	رفه	رسط	رما	ریج	قصج	قفز	قعر	قسط	قسط	فکا	صعج	عز	یخ	کا	رلد
	نح	شیا	رفز	رعا	رمح	رमे	ریز	قصر	قعج	قفا	قفط	قسه	قنه	قیز	صط	صا	عه	نا
	قلد	یخ	مه	سط	فه	قط	قلز	قنز	قفج	قفه	قعه	ره	رکه	رنج	رعز	رصح	شیز	ثط
	رکو	رس	ط	ما	سه	قه	قلج	قصه	قنط	قسا	قصا	قصط	رکط	رنز	رمنز	شکا	شنج	قب
	رک	قد	فو	ه	لز	قا	ریه	قله	قلط	قما	ریا	رز	ریط	رسا	شکه	شنز	رعو	رنح
	قمد	قی	رعد	قند	ا	رمج	قح	قز	قیبا	قیج	رلط	رله	رلا	رمز	شسا	رح	فح	رنب
	قمو	رن	رعب	قنو	قنب	ششط	شکج	رصه	رعه	صز	صه	فح	یخ	له	ققد	رو	ص	قیب
	رید	رمح	صب	رب	فح	عو	شنه	شیط	رصا	فط	عط	نط	لا	عح	رفب	رد	رع	قید
	ریب	قیو	رکب	شد	س	سب	رصح	شنا	شیه	عج	نه	کز	رصو	یخ	ع	رص	رکد	رمو
	قنبت	رمب	شکح	لو	لح	شکب	شک	مد	ثمز	مط	کج	مو	شید	شلیب	نب	ند	شو	رمد
	سب	شمس	د	و	شند	شنب	یب	ید	ثمو	یز	شمد	ک	کب	شلح	شلو	کح	ل	شل

لوح تسعة عشر في تسعة عشر وآخر الاعداد فيها ٣٦١ وجملتها ٦٥٣٤١ وكل صف من : tit.
صفوف ٣٤٣٩

رمج || (v. *loculum infra*, 13, 6) ۱۰۳ || قله (12, 7) ۲۴۹ || شمط (2, 9) ۲۳۷ || شلز (17, 12) ||
 رمد || *ut vid.* رنب (19, 3) ۲۲۳ || شكج (13, 5) ۱۷۵ || رعه (11, 5) ۳۴۳ || (14, 6) ||
ut vid. شغب (14, 1) ۲۴۷ || شبن (11, 2) ۲۴۷ || رعد (2, 2) ||

98	2	358	356	8	10	350	348	16	345	18	342	340	24	26	334	332	32	100
200	118	34	326	324	40	42	318	15	313	339	316	48	50	310	308	56	120	162
198	196	138	58	302	300	64	11	47	289	307	335	66	294	292	72	140	166	164
82	194	192	158	74	286	7	43	71	273	283	303	331	284	80	160	170	168	280
84	106	190	188	178	3	39	67	87	265	267	279	299	327	180	174	172	256	278
122	108	130	186	329	115	259	255	251	249	123	127	131	119	33	176	232	254	240
238	94	132	333	301	233	143	227	223	221	151	155	147	129	61	29	230	268	124
236	266	337	305	281	237	209	163	203	201	171	167	153	125	81	57	25	96	126
128	341	309	285	269	241	213	193	187	177	179	169	149	121	93	77	53	21	234
343	311	287	271	263	245	217	197	173	181	189	165	145	117	99	91	75	51	19
134	13	45	69	85	109	137	157	183	185	175	205	225	253	277	293	317	349	228
226	260	9	41	65	105	133	195	159	161	191	199	229	257	297	321	353	102	136
220	104	86	5	37	101	215	135	139	141	211	207	219	261	325	357	276	258	142
144	110	274	154	1	243	103	107	111	113	239	235	231	247	361	208	88	252	218
146	250	272	156	182	359	323	295	275	97	95	83	63	35	184	206	90	112	216
214	248	92	202	288	76	355	319	291	89	79	59	31	78	282	204	270	114	148
212	116	222	304	60	62	298	351	315	73	55	27	296	68	70	290	224	246	150
152	242	328	36	38	322	320	44	347	49	23	46	314	312	52	54	306	244	210
262	360	4	6	354	352	12	14	346	17	344	20	22	338	336	28	30	330	264

$$n = 19$$

(36) فامّا العمل في مربّعات الازواج فانّها على ثلاثة اقسام زوج
 وزوج فرد وزوج زوج وفرد.
 فاؤلّها مربّع الاربعة لانّ اؤلّ الازواج هو اثنان ولا يمكن فيه هذا
 اعنى ان يُجعل فيه اعداد وفقاً.

425

(37) فلنعمد الى «مربّع» الاربعة. فتجعل احدى واسطتيه في «بيت»
 وسطه والاخرى في زاويته [مثل مسرى الفيل في الشطرنج] وهى الثالثة
 «وهى البيت الثالث» منه على قطره. ثمّ العدد الذى يلي الواسطة
 الصغرى الى جانب الواسطة الكبرى عن يمينها وما يلي الواسطة الكبرى

(36) A: 17^v, 17-17^v, 20 (ed. ll. 806-809).

tit. A: الباب منها الثانى فى علم الاعداد الازواج :

424 *cod.* وفق [وفقاً 425] || *cod.* زوج الفرد [مربّع الاربعة 424

(37) A: 17^v, 20-17^v, 21 & 19^f, 1-19^f, 11 (ed. ll. 810-822).

426 *cod.* الذى [الذى 431] || *cod.* الواسط [*quasi* (post.) الواسطة 429] || *cod.* فلنعبده [فلنعمد 426

(Bordered even-order squares)

(§36. The three categories of even-order squares)

The treatment for squares of even (orders) depends on their three categories: evenly-even, evenly-odd and evenly-evenly-odd.¹¹²

The first (square of even order) is the square of four; indeed, the first even (order) is 2 and this is not possible for it, that is, to place in it numbers in magic arrangement.

9	7	14	4
6	12	1	15
3	13	8	10
16	2	11	5

5	10	15	4
11	8	1	14
2	13	12	7
16	3	6	9

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

$$n = 4$$

 (§37. Construction of the square of order 4)¹¹³

So let us turn our attention to the (square of) four.¹¹⁴ You place (the numbers as follows).¹¹⁵ One of its two median numbers (8) in (a cell in) its centre and the other (9) in its corner [like the bishop's move in chess], namely the third (corner), (which is the third cell) diagonally from it.¹¹⁶ Then, the number preceding the small median (7) next to the large median, on its right¹¹⁷, and that following the large median,

¹¹² 'Evenly-even' (ἄρτιάκις ἄρτιος, Nicomachos' *Introduction* I.8.4, or ἄρτιάκις ἄρτιος μόνον, Euclid's *Elements* IX, 32), thus (here) of the form $n = 8t$, t natural ≥ 1 , such as 8, 16, 24, 32, 40, 48, ...; 'evenly-odd' (ἀρτιοπέριτος, Nicomachos I.9.1, or ἄρτιάκις περισσός μόνον, Euclid *Elements* IX, 33), thus of the form $n = 4t + 2$, such as 6, 10, 14, 18, 22, 26, ...; 'evenly-evenly-odd' (περισσάρτιος, Nicomachos I.10.1), thus (here) of the form $n = 8t + 4$, such as 12, 20, 28, 36, 44, 52, The author will now examine the squares with these even orders, after dealing with the particular case $n = 4$.

¹¹³ Since there is no magic square of order 2, as just stated, the square of order 4 will be filled as a whole without considering its outer border separately. The description of the placing will begin with its two medians, thus 8 and 9.

¹¹⁴ We have added the first figure, to correspond with the subsequent instructions. The manuscript ($\mathcal{A}$) has the second (same, but turned around the ascending diagonal), used in all subsequent even-order squares. The third is also found in the manuscript but not mentioned in the text (probably an earlier reader's addition); it starts with the *larger* median in the centre.

¹¹⁵ The text being corrupt, all our additions to it are explicitly indicated.

¹¹⁶ From the text, it is clear that 'third' refers to a corner, not a cell. In what follows, the original text must have had the words (added by us) 'third cell diagonally' — their replacement by a chess move must be Arabic. These modifications may explain the present corruptedness of the text.

¹¹⁷ As seen by the reader (p. 19, *n.* 25). From here on the instructions are no longer applicable to the second figure.

[its complement¹¹⁸] (10), in the [bishop's cell] ⟨third cell diagonally⟩ as well, next to the small median. Then, the (next) number following the small median (6) underneath the large median, and that following the large (median), [its complement¹¹⁹] (11), in the [bishop's cell] ⟨third cell diagonally⟩. Then, the (next) number following the small median (5) in the lower right-hand corner and its complement (12) in the [bishop's cell] ⟨third cell diagonally⟩ from it; the (next) number following the small median (4) in the upper right-hand corner and its complement (13) in the [bishop's cell] ⟨third cell diagonally⟩ from it; the (next) number following the small median (3) in the cell next to (that of) the larger number which you have reached [thus 3]¹²⁰, and the complement of it, thus 14, in the [bishop's cell] ⟨third cell diagonally⟩ from it; 2 in the cell underneath (that of) 13, and 15 in the [bishop's cell] ⟨third cell diagonally⟩ from it; 1 underneath 14, and 16 in the [bishop's cell] ⟨third cell diagonally⟩ from it. This square is finished.

(Bordered squares of orders $n = 4k + 2$)

(§ 38. Filling the inner square)

As to the treatment for the evenly-odd square, it is as follows.¹²¹ You draw the square, and then write the two medians of the numbers in the central square, where you had put the two medians of the previous square, the subsequent (numbers) where were the corresponding subsequent (numbers) of these two medians, (and so on) until you have finished with the inner square [as you had done with the odd (orders)].¹²² You are then left with the outer (part of the) square and the remaining numbers.

(§ 39. Filling evenly-odd borders)

Once you have done that, you take the two terms you have reached;¹²³ you put the small one — which will always be even if the sequence (of the numbers) proceeds from 1 to the last by successive additions of 1 — on the left-hand side underneath the corner cell, and its complement on the

¹¹⁸ Complement of 7 and not of the 'large median'.

¹¹⁹ Again, complement of 6, in spite of the Arabic form.

¹²⁰ Interpolation, whether we leave it as it is or change it to 'thus 13'.

¹²¹ After considering the square of order 4, the author treats the case of the subsequent even order 6 and its category. Next (§§ 40–43) he will deal with the other even orders. In later treatises, the study of squares with orders $n = 4k$ (including both $n = 8t$ and $n = 8t + 4$) precedes that of evenly-odd ones ($n = 4k + 2$).

¹²² Just as, for odd-order squares, we filled first the inner square (see §§ 4–5) —but this last remark seems superfluous. We are then left with filling the border.

¹²³ That is, the next small number in descending sequence, and the next in ascending sequence for the large numbers; thus, 10 and 27 for order 6, 18 and 83 for the border of order 10 (how to fill the preceding border of order 8 will be taught in § 41).

تقابله [بازائه. ثمّ تأخذ العددين اللذين يليان هذين فتثبت القليل منهما وهو فرد في الجهة السفلى والكثير في الجهة العليا [التى تقابله] بازائه. > ثمّ تأخذ العددين اللذين يليان هذين وهما زوجان ابداً فتثبت القليل منهما في الجهة اليمنى والكثير في الجهة اليسرى بازائه. > ثمّ كذلك ابداً تجعل الأزواج من الاعداد القليلة للجهتين اليمنى واليسرى مرّةً لهذا ومرّةً لهذا إلا أنّ المبتدأ من اليسرى وتجعل الافراد من القليلة للجهتين < العليا > والسفلى مرّةً لهذا ومرّةً لهذا إلا أنّ المبتدأ من السفلى حتّى تنتهى الى العدد الزوج القليل الذى قبل العدد الزوج الموافق لاسم ضلع المربع [بعدد فرد] فى الجهة اليمنى وذلك يتّفق ابداً فى البيت الذى هو تمام نصف الضلع.

فاذا اثبتته هناك فاثبت العدد الفرد القليل الذى يليه فى هذه الجهة ايضاً اعنى اليمنى وبازائه فى اليسرى ما يقابله. ثمّ اثبت العدد القليل الذى يتلوه وهو ما يوافق اسم الضلع [اعنى ان كان اسم الضلع ستة فعدد ستة وان كان عشرة فعدد عشرة] فى الزاوية اليسرى العليا وبازائه قطعاً فى الزاوية اليمنى ما يقابله. ثمّ اثبت العدد الفرد القليل الذى يتلو هذا العدد فى الزاوية العليا اليمنى وبازائه قطعاً فى الزاوية اليسرى السفلى

post [فى الجهة العليا التى تقابله بازائه 452 || cod. الكثير] والكثير 452 || cod. اخرها [اخيرها 449
١٠] عشرة 464 || cod. ٦ [ستة 464 || cod. ٦ [ستة 463 || hab. cod. (l. 452) وهو فرد
471 || cod. جعلت] جعلت 468 || cod. يقابل quasi [يقابله 467 || cod. ١٠] عشرة 464 || cod.

right-hand side [facing it (= this cell)], opposite to it (= the number). Next you take the two numbers following them;¹²⁴ you put the small one, which is odd, on the lower side, the large one on the upper side [facing it], opposite to it. (Next you take the two numbers following them, which will always be even, and you put the small one on the right-hand side and the large one opposite to it on the left-hand side.)¹²⁵ Then you continue placing in this way the small even numbers alternately on the right and on the left — but beginning on the left — and you place the small odd (numbers) alternately on the top and on the bottom — but beginning on the bottom — until you attain, on the right-hand side, the small even number which precedes the even number corresponding to the denomination of the side of the square¹²⁶ [with an odd number]¹²⁷, this occurring always in the cell completing half of the side.

6	36	2	34	28	5
10	15	20	25	14	27
29	21	18	11	24	8
30	12	23	22	17	7
4	26	13	16	19	33
32	1	35	3	9	31

$n = 6$

10	100	2	98	5	94	88	15	84	9
18	26	74	73	72	71	31	32	25	83
85	77	38	68	34	66	60	37	24	16
14	23	42	47	52	57	46	59	78	87
89	79	61	53	50	43	56	40	22	12
90	21	62	44	55	54	49	39	80	11
8	20	36	58	45	48	51	65	81	93
95	82	64	33	67	35	41	63	19	6
4	76	27	28	29	30	70	69	75	97
92	1	99	3	96	7	13	86	17	91

$n = 10$

Once you have written it there, put the next small odd number again on this same side, thus the right-hand one, and opposite to it on the left its complement. Put then the preceding small number, namely that corresponding to the denomination of the side [thus if the denomination of the side is 6, it will be the number 6; if it is 10, it will be the number 10]¹²⁸ in the upper left-hand corner, and opposite to it diagonally, in the (lower) right-hand corner, its complement. Put then the small odd number preceding this number in the upper right-hand corner, and opposite

¹²⁴ The next in each sequence, thus 9 and 28, and 17 and 84, respectively.

¹²⁵ This addition, at least, seems necessary.

¹²⁶ The even number preceding the order, thus 8 and 12. The expression ‘corresponding to the denomination’ (*al-muwāfīk li-ism*) must be the Greek ὁμώνυμος.

¹²⁷ This gloss may have originally referred to ‘beginning on the bottom’ above.

¹²⁸ Hardly the author’s since he would (presumably) have put it before, at the first occurrence of ‘denomination’.

to it diagonally, in the lower left-hand corner, the number which is its complement.

Once you have done that, you place the remaining small even numbers alternately on the right and left sides, beginning on the left, until you attain 4. Then you put opposite to each number its complement. This results in the completion of the right-hand and left-hand cells. You place (then) the ⟨remaining⟩ small odd numbers alternately on the top and on the bottom, beginning on the bottom, and you put opposite to each number its complement, until you reach 3. Once you have done this, you put 3 on the bottom, and opposite to it on the top its complement; 2 on the top, and opposite to it on the bottom its complement; 1 on the bottom, and opposite to it on the top its complement. In this way you will have finished what you wanted.

(Bordered squares of orders $n = 4k$)

(§40. Filling the inner square)

As to the treatment for the evenly-even and evenly-evenly-odd, it is as follows. You put the two medians and the numbers next to them in the square inside these squares until you complete it [as we have explained for the odd and evenly-odd].¹²⁹

(§41. Filling the inner part of the horizontal rows for the square of order 8)

If your treatment is for the square of eight, you take the numbers attained.¹³⁰ Put two consecutive small (numbers) on the top, excluding the corner cell; and you put their complements opposite to them on the bottom. Then, put four consecutive small (numbers) on the bottom and, opposite to them on the top, their complements.¹³¹

8	56	55	54	53	13	14	7
59	20	50	16	48	42	19	6
5	24	29	34	39	28	41	60
61	43	35	32	25	38	22	4
3	44	26	37	36	31	21	62
2	18	40	27	30	33	47	63
64	46	15	49	17	23	45	1
58	9	10	11	12	52	51	57

$n = 8$

¹²⁹ In §§4–5 and 38. But, as in §38 ($n. 122$), this reference seems superfluous.

¹³⁰ Thus we shall use the descending sequence from 14 and ascending from 51.

¹³¹ How to fill the vertical rows will be taught later on (§43).

(42) وان كان عملك لاكثر من ثمانية فان يحصل لك من الايات في كلّ صف بعد ايات الزوايا عشرة ايات واما اربعة عشر بيتاً واما ثمانية عشر بيتاً وكذلك ابداً بزيادة اربعة اربعة [على ما جعلت في مربع ثمانية]. واعلم انّ كلّ اربعة اعداد على النسق تقابل اربعة اعداد فهي اذا اثبت من الاربعة القليلة الاول منها والاخر في جهة ومن الاربعة الكثيرة الاوسطين في هذه الجهة واثبت بازائها في الجهة الاخرى ما يقابلها من الاعداد فعُدلت احدى الجهتين بالاخري. فان كان الذى يبقى من
 490 (الايات) بعد ايات الزوايا عشرة ايات تعدّل اربعة منها من فوق باربعة توازيها من اسفل باربعة اعداد من الاعداد التى تنتهى اليها بعد حشو المربع الداخل وما يقابلها. وكذلك ان بقى اربعة عشر فعُدلت ثمانية بثمانية. وكذلك ابداً حتّى يبقى لك ستّة ايات من فوق وستّة من اسفل. فاذا فعلت ذلك فاثبت عددين من الاعداد القليلة التى تنتهى اليها
 495 على النسق في الجهة العليا وبازائهما فى السفلى ما يقابلهما. ثمّ اثبت اربعة من القليلة التى تنتهى اليها على النسق فى الجهة السفلى وبازائها فى العليا ما يقابلها.

(42) A: 19^v, 3-19^v, 17 (ed. ll. 863-877).

١٠ عشرة 485 || *cod.* [فان 484 || *cod.* ثمنيه 484
 ٤ ٤ || *cod.* [اربعة اربعة 486 || *cod.* ١٨ ثمانية عشر 485 || *cod.* ١٤ [اربعة عشر 485 || *cod.*
 ١٠ عشرة 491 || *add.* [الكثيرة 488 || *cod.* الجبه 488 || *cod.* ٨ ثمانية 486
 [فعُدلت 493 || *cod.* ١٤ [اربعة عشر 493 || *cod.* يقابله [يقابلها 493 || *cod.* ٤ [اربعة 491
 (sic) ٦ [وستّة 494 || *cod.* ٦ ستّة 494 || *cod.* ٨ سمنيه [ثمانية بثمانية 493-4 || *cod.* وعدلت
 497 || *cod.* ٤ [اربعة 497 || *cod.* يقابلها [يقابلها 496 || *cod.* وبارايها [وبازائهما 496 || *cod.*
 [السفلى] *corr. ex* سفلى *cod.*

(§42. Filling the inner part of the horizontal rows for higher orders)

If your treatment is for (an order) higher than eight,¹³² you have as (quantity of) cells in each row, excepting the corner cells, ten cells, or fourteen cells, or eighteen cells, and so on always by successive additions of 4 [according to what you have placed in the square of 8]¹³³. Now you are to know that, for any group of four consecutive numbers and four complementary numbers, if one puts on one side the first and the last of the four small numbers and, on this same side, the two middle of the four large ones, and one puts opposite to them on the other side the numbers which are their complements, the two sides will be equalized.¹³⁴ Thus if, dismissing the corner cells, there remain ten cells, you will equalize four on the top with the four facing them on the bottom by means of four (consecutive) numbers of those you have reached after filling the inner square and their complements.¹³⁵ Likewise, if fourteen (cells) remain, you will equalize eight with eight. And always like that until you are left with six (empty) cells on the top and six on the bottom.

12	132	131	130	129	17	18	19	125	124	22	11
135	32	106	37	110	116	27	120	24	122	31	10
9	40	48	96	95	94	93	53	54	47	105	136
137	107	99	60	82	88	56	90	59	46	38	8
7	36	45	64	69	74	79	68	81	100	109	138
6	111	101	83	75	72	65	78	62	44	34	139
140	112	43	84	66	77	76	71	61	102	33	5
141	30	42	58	80	67	70	73	87	103	115	4
3	117	104	86	63	57	89	55	85	41	28	142
2	26	98	49	50	51	52	92	91	97	119	143
144	114	39	108	35	29	118	25	121	23	113	1
134	13	14	15	16	128	127	126	20	21	123	133

$$n = 12$$

Once you have done that, put, of the small numbers you have reached, two consecutive numbers on the top and, opposite to them on the bottom, their complements. Then put, of the small numbers you have reached,

¹³² Considering thus horizontal rows for both orders $n = 8t$, $t \neq 1$, and $n = 8t + 4$.

¹³³ Misplaced gloss (see below).

¹³⁴ This 'neutral' arrangement has been already mentioned for a sequence of even numbers (§20, text corrupt), and used several times since (§§31, 33, 34).

¹³⁵ For order 12, we begin with 22 and 123, respectively. The numbers left will be consecutive since that is what we have placed so far.

(43) فاذا فعلت ذلك فاثبت العدد القليل الذى تنتهى اليه وهو [زوج
 500 ابداً] فى هذا النسق موافق لاسم الضلع فى بيت الزاوية اليسرى العليا
 وبازائه قطرًا فى الزاوية اليمنى ما يقابله. ثم اثبت القليل الذى يتلوه وهو
 فرد فى الزاوية اليمنى العليا وبازائه قطرًا فى الزاوية اليسرى ما يقابله. ثم
 اثبت القليل الذى يتلو هذا العدد فى الجهة اليمنى وما يقابله فى الجهة
 اليسرى والعدد القليل الذى يلى هذا العدد فى الجهة اليسرى وما يقابله
 505 فى الجهة اليمنى. فاذا فعلت هذا فقد عدلت هاتين الجهتين ايضًا.
 ويكون ما يبقى لك منهما اربعة ابيات تقابل اربعة او ثمانية تقابل
 ثمانية او اثنا عشر تقابل اثنى عشر وكذلك بزيادة اربعة اربعة فتعدل كل
 اربعة ابيات منها بالاربعة التى تقابلها كما وصفنا وهو ان تثبت الاول
 من القليلة والاخير فى جهة والاوسطين <من الكثيرة> فى هذه الجهة وما
 510 يقابلها فى الجهة الاخرى بازائها حتى تأتى عليها كلها.

(43) A: 19^v, 17-20^f, 8 (ed. ll. 878-890).

507 || ٨ ثمانية 506 || cod. و [او 506 || cod.] (post.) اربعة 506 || cod. ٤ [اربعة 506
 507 || ١٢ تقابل ١٢ in textu, et add. in marg.: ١٨ [ثمانية او اثنا عشر تقابل اثنى عشر
 cod. ٤ [اربعة 508 || cod. ٤ ٤ [اربعة اربعة

four consecutive ones on the bottom and, opposite to them on the top, their complements.¹³⁶

(§43. Filling the corners and the vertical rows)

Once you have done that, put the small number you have reached — which [is always even]¹³⁷, for this succession, corresponds to the denomination of the side — in the upper left-hand corner cell and, opposite to it diagonally, in the right-hand corner, its complement. Put then the subsequent small number, which is odd, in the upper right-hand corner and, opposite to it diagonally, in the left-hand corner, its complement. Put then the subsequent small number on the right-hand side and its complement on the left-hand side, and the subsequent small number on the left-hand side and its complement on the right-hand side. Once you have done this, you will have equalized these two sides as well.¹³⁸

What you are left with (as empty cells) in these two (vertical rows) is four cells facing four,¹³⁹ or eight facing eight, or twelve facing twelve, and so on with successive additions of 4. You will then equalize each group of four cells with their four opposite as we have explained,¹⁴⁰ that is, putting (for each group) on one side the first and the last small and the two large middle, and their complements opposite to them on the other side, until you have finished with all (empty cells).

¹³⁶ The above gloss fits here.

¹³⁷ What comes next makes this information superfluous.

¹³⁸ Indeed, the four cells thus filled in each vertical row make their sum due.

¹³⁹ Case $n = 8$, thus here completed.

¹⁴⁰ §42. See also *n.* 134.

ه	ك	لد	ب	لو	و
كز	يد	كه	ك	يه	ى
ح	كد	يا	يح	كا	كط
ز	نز	كب	كج	يب	ل
لج	يط	يو	يج	كو	د
لا	ط	ج	له	ا	لب

ح	يح	يب	ا
ى	ج	و	يه
ه	يو	ط	د
يا	ب	ز	يد

د	يه	ى	ه
يد	ا	ح	يا
ز	يب	يح	ب
ط	و	ج	يو

ط	فد	يه	فخ	صد	ه	صع	ب	ق	ى
فخ	كه	لب	لا	عا	عب	عج	عد	كو	يح
يو	كد	لز	س	سو	لد	سج	لح	عز	فه
فز	عح	نط	مو	نر	نب	مز	مب	كج	يد
يب	كب	م	نو	مخ	ن	نج	سا	عط	فط
يا	ف	لط	مط	ند	نه	مد	سب	كا	ص
صعج	فا	سه	نا	مح	مه	مخ	لو	ك	ح
و	يط	سج	ما	له	سز	لج	سد	فب	صه
صز	عه	سط	ع	ل	كط	كح	كز	عو	د
صا	يز	فو	يح	ز	صو	ج	صط	ا	صب

(4 × 4) $\mathcal{A}$: 25^r (ed. p. 312).

tit. : ٤ فى ٤

Eandem fig. præb. D, fol. 63^v, cum tit. : ٣٤ لوح اربعة منسوقة على الواحد كل صف منه

(4 × 4) $\mathcal{A}$: 25^r (ed. p. 312).

tit. : ٤ فى ٤

(6 × 6) $\mathcal{A}$: 25^v (ed. p. 312).

tit. : ٦ فى ٦

(10 × 10) $\mathcal{A}$: 25^r (ed. p. 312).

tit. : ١٠ فى ١٠

نح. $corr. ex$ [(4, 6) نح || قط (1, 6) فط

5	10	15	4
11	8	1	14
2	13	12	7
16	3	6	9

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

$n = 4$

6	36	2	34	28	5
10	15	20	25	14	27
29	21	18	11	24	8
30	12	23	22	17	7
4	26	13	16	19	33
32	1	35	3	9	31

$n = 6$

10	100	2	98	5	94	88	15	84	9
18	26	74	73	72	71	31	32	25	83
85	77	38	68	34	66	60	37	24	16
14	23	42	47	52	57	46	59	78	87
89	79	61	53	50	43	56	40	22	12
90	21	62	44	55	54	49	39	80	11
8	20	36	58	45	48	51	65	81	93
95	82	64	33	67	35	41	63	19	6
4	76	27	28	29	30	70	69	75	97
92	1	99	3	96	7	13	86	17	91

$n = 10$

ح	نط	ه	سا	ج	ب	سد	نح
نو	ك	كد	مچ	مد	مچ	مو	ط
نه	ن	كط	له	كو	م	يه	ي
ند	يو	لد	لب	لز	كز	مط	يا
نچ	مچ	لط	كه	لو	ل	يز	يب
يچ	مب	كح	لح	لا	لج	كج	نب
يد	يط	ما	كب	كا	مز	مه	نا
ز	و	س	د	سب	سج	ا	نز

ز	يد	يچ	نچ	ند	نه	نو	ح
و	يط	مب	مچ	يو	ن	ك	نط
س	ما	كح	لط	لد	كط	كد	ه
د	كب	لح	كه	لب	له	مچ	سا
سب	كا	لا	لو	لز	كو	مد	ج
يچ	مز	لج	ل	كز	م	يچ	ب
ا	مه	كج	يز	مط	يه	مو	سد
نز	نا	نب	يب	يا	ي	ط	نح

يا	كب	فكد	فكه	يط	يچ	يز	فكط	قل	قلا	قلب	يب
ي	لا	فكب	كد	كك	كز	قيو	قي	لز	قو	لب	قله
قلو	قه	مز	ند	نچ	صمچ	صد	صه	صو	مح	م	ط
ح	لح	مو	نط	ص	نو	فخ	فب	س	صط	قز	قلز
قلح	قط	ق	فا	سج	عط	عد	سط	سد	مه	لو	ز
قلط	لد	مد	سب	عح	سه	عب	عه	فخ	قا	قيا	و
ه	لج	قب	سا	عا	عو	عز	سو	فد	مچ	قيب	قم
د	قيه	فج	فز	عج	ع	سز	ف	نچ	مب	ل	قما
قنب	كح	ما	فه	نه	فط	نز	سج	فو	قد	قيز	ج
قمح	قبط	صز	صا	صب	نپ	نا	ن	مط	صمچ	كو	ب
ا	قيج	كج	ككا	كه	قيح	كط	له	فخ	لط	قيد	قمد
قلج	فكج	كا	ك	فكو	فكز	فكح	يو	يه	يد	يچ	قلد

(8 × 8) $\mathcal{A}$: 25^v (ed. p. 312).

tit. : ۸ فی ۸

Alteram fig. præb. cod.

(12 × 12) $\mathcal{A}$: 26^r (ed. p. 313).

tit. : ۱۲ فی ۱۲

لوح اثني عشر منسوق على الواحد وآخر الاعداد فيها ۱۴۴ : cum tit. : ۱۰۴۴۰ ومبلغ كل صف منه ۸۷۰

|| ما [(3, 7)] قا || قح [(12, 8)] قلح || و [(10, 8)] ق || سج [(8, 8)] سج || فكر [(1, 9)] قلز || فك [(10, 12)] فكد
|| ملر [(7, 1)] فكز || قلح [(6, 1)] فكح || مد [(3, 4)] قد || فج [(10, 5)] قج || س [(6, 5)] سز || قد [(4, 6)] فد
فلج [(11, 1)] فكج.

8	56	55	54	53	13	14	7
59	20	50	16	48	42	19	6
5	24	29	34	39	28	41	60
61	43	35	32	25	38	22	4
3	44	26	37	36	31	21	62
2	18	40	27	30	33	47	63
64	46	15	49	17	23	45	1
58	9	10	11	12	52	51	57

$$n = 8^{141}$$

12	132	131	130	129	17	18	19	125	124	22	11
135	32	106	37	110	116	27	120	24	122	31	10
9	40	48	96	95	94	93	53	54	47	105	136
137	107	99	60	82	88	56	90	59	46	38	8
7	36	45	64	69	74	79	68	81	100	109	138
6	111	101	83	75	72	65	78	62	44	34	139
140	112	43	84	66	77	76	71	61	102	33	5
141	30	42	58	80	67	70	73	87	103	115	4
3	117	104	86	63	57	89	55	85	41	28	142
2	26	98	49	50	51	52	92	91	97	119	143
144	114	39	108	35	29	118	25	121	23	113	1
134	13	14	15	16	128	127	126	20	21	123	133

$$n = 12$$

¹⁴¹ In MS. $\mathcal{A}$ this figure is rotated (thus 8, 59, ..., 58 at the top).

(44) وأما تقسيم الأزواج فهو أن ترسم المربع الكبير الذي يكون من ضرب عدد زوج في مثله ثم يُقسم داخله بمربعات متساوية الاضلاع نحو أن يُقسم بأربعة مربعات ضلع كل مربع منها مثل نصف ضلع المربع الكبير أو يُقسم بتسعة مربعات ضلع كل واحد منها مثل ثلث ضلع المربع الكبير أو يُقسم بستة عشر مربعًا ضلع كل مربع منها مثل ربع ضلع المربع الكبير (أو يُقسم بخمسة وعشرين مربعًا ضلع كل واحد منها مثل خمس ضلع المربع الكبير). أو يُقسم المربع الكبير بمربعات مختلفة الاضلاع نحو أن يكون بعضها من ضرب ثمانية في ثمانية وبعضها من ضرب ستة في ستة وبعضها من ضرب أربعة في أربعة.

ثم ترتب الأعداد في جميع أبيات المربع الكبير من الواحد إلى نهاية حملتها فيكون المربع متى أجمل من جميع جهاته اتفقت الجمل ومتى أجمل مربع من تلك المربعات التي في داخله على حدته اتفقت جملة من جميع الجهات. وهذا يتهيأ في أعداد زوج الزوج وفي أعداد زوج الزوج

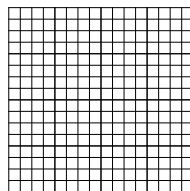
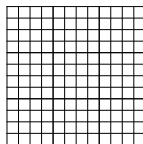
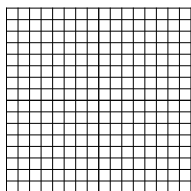
(44) A: 20^r, 9-20^r, 18 (ed. ll. 891-901); D: 99^r, 7-99^r, 13.

مع الملق من: *post quae*, المقالة السادسة في الوفق المتميز: *D tit.* in summa pag. praeb. D: 99^r, 7-99^r, 13. المربعات الصغار والكبار والحلق والصليب والفواصل بعضها مع بعض وهذه كلها تقع في التقسيم (...)

511 || D ومعنى التقسيم هو أن A. وأما قسم الأزواج فهو أن 511 || A om. [بمربعات ... يُقسم 512-3 || D نقسم] يُقسم 512 || A الأكبر [الكبير 511 || D نرسم] نرسم A, كل عدد مربع [ضلع كل مربع 513 || AD باربع [بأربعة 513 || D متساويات [متساوية 512 || 514 || D om. [المربع 513 || A om. (post.) ضلع 513 || D ربع [نصف 513 || D كل ضلع ملت [ثلث 514 || AD الواحد [كل واحد 514 || D تسع, A, سبع [بتسعة 514 || A الكثير [الكبير l. المربع الكبير post] أو يُقسم ... المربع الأكبر 515-6 || A الأكبر [المربع الكبير 514-5 || D 517-8 || A om. (homoeotel.) D 515 || A, om. [نصف ربع 515 || D من 518 || D om. [المربع الكبير 517 || A أو تقسم في مربع واحد ومربعين [الكبير ... نحو أن ستة في 518-9 || A ٨ في ٨ [ثمانية في ثمانية 518 || D om. [ضرب ثمانية ... وبعضها من ضرب [ترتب 520 || A ٤ في ٤ [أربعة في أربعة 519 || D om. [من ضرب 519 || A ٦ في ٦ [ستة انتهى [يتهيأ 523 || D وهذه [وهذا 523 || D نهايته, A, نهايته حملته [نهاية حملتها 520-1 || D رتب انتهى [يتهيأ 523 || D وهذه [وهذا 523 || D وفي أعداد 523 || A أعدادا [pr.] أعداد 523 || D

(Composite even-order squares)¹⁴² (§44. Composite squares for orders divisible by 4)¹⁴³

As for the division of the even (order squares), it proceeds as follows. One draws the main square, which arises from the multiplication of an even number by itself; then its inner part is divided into squares all equal.¹⁴⁴ For instance, it is divided into four squares with the side of each equal to half the side of the main square;¹⁴⁵ or it is divided into nine squares with the side of each equal to a third of the side of the main square;¹⁴⁶ or it is divided into sixteen squares with the side of each equal to a fourth of the side of the main square;¹⁴⁷ (or it is divided into twenty-five squares with the side of each equal to a fifth of the side of the main square.)¹⁴⁸



Or the main square is divided into unequal squares.¹⁴⁹ For instance, some of these (subsquares) will have the size 8 by 8 and others the size 6 by 6 and others the size 4 by 4.¹⁵⁰

Then you will arrange the numbers from 1 to the end of their quantity in all the cells of the main square in such manner that in the (main) square the sums are everywhere the same, and that in each inner square consid-

¹⁴² Subject of §§44–54. All subsquares must be of even order and filled with pairs of complements (filling described in §48). In our transcription of the examples (pp. 127–161) the squares will have the left to right orientation.

¹⁴³ Thus of the form $n = 4k$, $k \geq 2$ (that is, both orders $n = 8t$ and $n = 8t + 4$, see §36). Even orders divisible by 2 only will be considered in the next paragraph.

¹⁴⁴ ‘Squares having equal sides’, in the text; perhaps translating *ισόπλευρα τετράγωνα* (although *τετράγωνον* has mostly our sense of ‘square’).

¹⁴⁵ Main order $n = 4t$, $t \geq 2$. Examples given for $n = 12$ (p. 127) and $n = 16$ (p. 131).

¹⁴⁶ Main order $n = 6t$, t even. Example for $n = 12$ (p. 129).

¹⁴⁷ Main order $n = 8t$, $t \geq 2$. Example for $n = 16$ (p. 133).

¹⁴⁸ Main order $n = 10t$, t even. We have added this last case, omitted in the two manuscripts; for the text constructs one example of such a square (p. 137, $n = 20$).

¹⁴⁹ ‘Squares having unequal sides’, in the text; perhaps translating *ἀνισόπλευρα τετράγωνα*.

¹⁵⁰ Examples below, pp. 143 ($n = 16$), 147 & 149 ($n = 20$).

والفرد.

(45) فأما اعداد زوج الفرد فلا يتهياً ان يُقسم المربع ارباعاً > ثم ترتب
525 (الاعداد فيها) فيودى الى هذه الشريطة [لأن جملة العدد الذى يقع فيه
لا يتهياً ان يكون له رُبع صحيح من غير كسر وليس من الشريطة ان
يُثبت فى شىء من هذه المربعات كسر فلذلك يتعذر هذا فى زوج الفرد
فى التنصيف خاصة]. فأما ما كان من عدد زوج الفرد الذى له ثلث او
جزء من الاجزاء غير النصف مثل الثمانية عشر والثلثين وما اشبههما فإنه
530 يتهياً ان يُقسم على الجزء الذى يوجد فيه [غير النصف] من الاجزاء التى
يكون عددها زوجاً [ونجاوز الاثنين لأن الاثنين لا حيلة فيهما كما
قدّمنا] ثم ترتب الاعداد فيه فيودى الى ما ذكرنا.
فان مربع الثمانية عشر يُقسم بتسعة مربعات كلّ مربع منها من ضرب
535 ستة فى ستة > ثم ترتب الاعداد فيها > فتكون الجمل التى فى الكبير
متفقة وفى كلّ واحد من هذه المربعات متفقة ايضاً. ومربع الثلثين يُقسم
بتسعة مربعات كلّ واحد من ضرب عشرة فى عشرة ويُقسم ايضاً بخمسة
وعشرين مربعاً كلّ واحد من ضرب ستة فى ستة [ويُقسم ايضاً فى اربع

(45) A: 20^r, 18-20^v, 15 (ed. ll. 902-919); D: 99^r, 13-99^r, 24.

له 527 || A iter. العدد 526 || D ويودى A, ويودى 526 || D منها [المربع 525
> hoc] تعذر [يتعذر 528 || A فكدلك 528 || AD فى (post.) من 527 || D اربع [رُبع
D و [او 529 || D التى لها [الذى له 529 || AD اعداد [عدد 529 || D
اشبه [اشبهما 530 || A او ما [وما 530 || A او 30 || A والثلثين 530 || A 18 || A [الثمانية عشر 530 ||
فى الجهة اليمنى التى post hoc (pr.) الاثنين 532 || A ut vid. وسحاور [ونجاوز 532 || A ذلك
533 || A فيها [فيهما 532 || D حبله [حيلة 532 || A add. et del. (v. ll. 450-1) تقابله بازائه
A, [بتسعة 534 || A 18 || A [الثمانية عشر 534 || D om. D [مربع 534 || D فترنا قبل [قدّمنا
الجمله 535 || A 6 فى 6 || A ستة فى ستة 535 || A om. D [كلّ مربع 534 || D بتسع
A, ومربع 30 || A [ومربع الثلثين 536 || A ممقق (post.) متفقة 536 || A om. [التى 535 || A
10 فى [عشرة فى عشرة 537 || D تسع A, 9 [بتسعة 537 || A سقس [يُقسم 536 || D والثلثون
538 || A 6 فى 6 || A ستة فى ستة 538 || A مربع [مربعاً 538 || A وسقس [ويُقسم 537 || A 10

ered by itself the sums are everywhere the same.¹⁵¹ Such (constructions) are (always) possible for the evenly-even and evenly-evenly-odd orders.¹⁵²

(§ 45. Division of squares of orders $n = 4k + 2$, $k \geq 2$, into equal parts)

As for the evenly-odd orders, it is not possible to divide the (main) square into four parts, then to arrange the numbers in them in such a way as to satisfy the above condition.¹⁵³ [Indeed, it is absolutely impossible for the number in question to have an integral fourth, without fraction; now, as a rule, a fraction could not possibly be put in these squares. That is why this (kind of division) is impossible for evenly-odd (orders, that is) for the specific case of division (of this order) by two.] As for an evenly-odd order having a third or some part other than a half, as (for the orders) eighteen or thirty or the like, it will be possible to divide it according to the part found in it [other than the half] which corresponds to an even order [we leave out 2 which, as noted by us before, offers no possibility].¹⁵⁴ Arranging then there the numbers, it will lead to what we have explained.

Thus, the square of eighteen is divisible into nine squares, each with size 6 by 6, where the numbers will be arranged in such a way that the sums in the main (square) are the same and also (those) in each of these (smaller) squares.¹⁵⁵ The square of thirty is divisible into nine squares, each with size 10 by 10, and also into twenty-five squares, each with size 6 by 6, [and also, in the four corners, into four squares, each with size 12 by 12, separated by nine squares, each with size 6 by 6,]¹⁵⁶ where the numbers will then be arranged in such a way that the sums be everywhere the same, in the main (square) as well as in each of the small ones.

¹⁵¹ How to arrange the numbers will be explained in § 48. Thus the main square as well as the subsquares all display the magic property. In the case of subsquares of different sizes, the arrangement of the subsquares within the main square must be such that the latter's diagonals make the magic sum. The somewhat odd 'everywhere' (*min jamā' al-jihāt*; cf. pp. 30–31) might translate πάντη.

¹⁵² Thus for orders of the general form $n = 4k$, $k \geq 2$ (above, *n.* 143). The possibility for the remaining category, that of evenly-odd orders, will be now examined.

¹⁵³ The four subsquares will be of odd order, whereas filling a subsquare exclusively with *pairs* of complements requires its order to be even, thus the order of the main square to be divisible by 4. This statement was misunderstood by an early reader.

¹⁵⁴ The subsquares, being of even orders, may then be filled with pairs of complements. The glosses are repetitive.

¹⁵⁵ Example below, p. 135.

¹⁵⁶ This ought to be in the next paragraph. Furthermore, it must be interpolated: the orders of the subsquares considered here do not exceed 10 (see p. 149).

زوايا باربعة مربعات يكون كل واحد منها من ضرب اثني عشر في اثني
 540 عشر وتُفصل بتسعة مربعات كل مربع (منها) من ضرب ستة في ستة [
 ثم ترتب الاعداد فيها فتتفق الجمل في الكبير من جميع الجهات وفي كل
 واحد من الصغار كذلك ايضاً.

(46) ويمكن ايضاً في هذين العددين وغيرهما من اعداد زوج الفرد ان
 يُقسم على اجزاء مختلفة وترسم فيها الاعداد فتودى الى اتفاق الجمل على
 545 الشريطة.

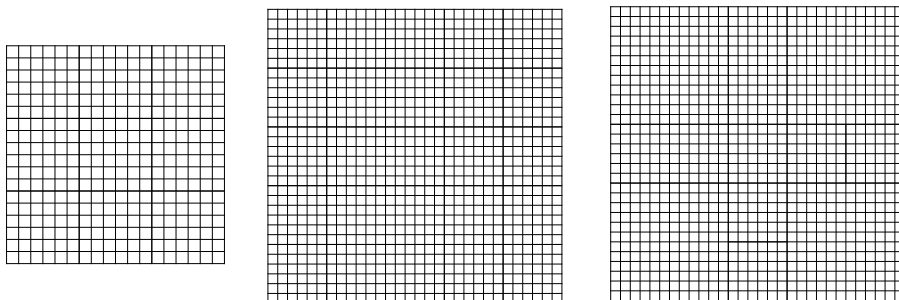
فان العشرة يمكن ان يكون في اربع زواياها اربعة مربعات كل واحد
 منها من ضرب اربعة في اربعة اذا اجمل كل واحد منها من جميع جهاته
 اتفقت جملة واذا اجمل كل المربع من جميع جهاته اتفقت جملة. ويمكن
 ايضاً في الاربعة عشر ان يكون في زواياه الاربعة اربعة مربعات كل
 واحد منها من ضرب ستة في ستة اذا اجمل كل مربع منها من جميع
 550 جهاته اتفقت جملة واذا اجمل المربع الكبير من جميع جهاته اتفقت جملة.
 ويتبيّن ايضاً ان يكون في وسط هذا المربع مربع من ضرب ستة في ستة
 ويحيط به ثمانية مربعات كل واحد منها من ضرب اربعة في اربعة اذا
 اجمل كل واحد منها على حدته اتفقت جملة واذا اجمل كل المربع اتفقت

|| A باربع [باربعة 539 || D في زواياها, A اربع زوايا [في اربع زوايا 538-9 || A وينقسم [وينقسم
 ونصلب [وتُفصل 540 || A ١٢ في ١٢ [اثني عشر في اثني عشر 539-40 || A ويكون [يكون 539
 541 || D ترتب [ترتب 541 || A ٦ في ٦ [ستة في ستة 540 || D بخمسة, A ٥ [بتسعة 540 || D
 A. منها [الكبير add.

(46) A: 20°, 15-21°, 8 (ed. ll. 920-933); D: 99°, 24-99°, 11.

tit. D : فصل (iter. ante ويمكن, l. 548)

544 || A pr. scr. et del. [الفرد 543 || A وفي غيرهما [وغيرهما 543 || A om. [ايضاً 543
 ٤ في [اربعة في اربعة 547 || A اربع [اربعة 546 || A قد يمكن [يمكن 546 || D فيودى [فتودى
 الاربعة 549 || D ومجمله [pr.) جملة 548 || A الجهات [جهاته 547 || D حمل [اجمل 547 || A ٤
 ٦ في [ستة في ستة 550 || A ٤ [اربعة 549 || D اربع زوايا [زواياه الاربعة 549 || A ١٤ [عشر
 ستة في 552 || A om. [جملة واذا ... جملة 551 || D حمل [اجمل 550 || A فاذا [اذا 550 || A ٦
 554-5 || A ٤ في ٤ [اربعة في اربعة 553 || D om. [منها 553 || A ٨ [ثمانية 553 || A ٦ في ٦ [ستة



(§ 46. Division of squares of orders $n = 4k + 2$, $k \geq 2$, into unequal parts)

It is (also) possible with these two orders¹⁵⁷ as well as with other evenly-odd orders to carry out a division into different parts and to place in them the numbers so as to be led to the constancy of the sums in compliance with the condition.¹⁵⁸

Thus (for the square of) ten it is possible to have in its four corners four squares each with size 4 by 4 displaying each equal sums everywhere, with the sums in the whole square being (themselves also) everywhere the same.¹⁵⁹ It is likewise possible for the (square of) fourteen to have in its four corners four squares of size 6 by 6, each displaying sums everywhere the same, with the sums in the main square being (themselves also) everywhere the same;¹⁶⁰ it is moreover possible to have in the centre of this square a square of size 6 by 6 surrounded by eight squares of size 4 by 4, each displaying individually same sums, with the sums in the whole square being the same.¹⁶¹ It is again possible for the square of (order) eighteen to have in its centre a square of size 10 by 10 surrounded by twelve squares of size 4 by 4, each displaying also same sums;¹⁶² it is moreover possible to have in its four corners four squares of size 8 by 8 displaying equal sums.¹⁶³ And the same for all such squares.

¹⁵⁷ Namely $n = 18$ and $n = 30$.

¹⁵⁸ Namely, constancy of the sums in the main square and in its various parts, as will be specified below.

¹⁵⁹ Example below, p. 155. How to fill the remaining central cross will be explained in detail further on (§§ 50–54).

¹⁶⁰ Example below, p. 157.

¹⁶¹ Example below, p. 139 — with, as seen here in the illustration, strips separating the 4×4 squares.

¹⁶² Example below, p. 145, with strips separating the 4×4 squares.

¹⁶³ Examples below, with a central cross and four 8×8 squares or sixteen 4×4 squares, see pp. 159 and 161. Although the square of order 30 is mentioned at the beginning, there is no example for it here (see p. 107 & $n. 156$). The author may have confined himself to the examples actually constructed by him.

555 جملة. ويمكن في مربع ثمانية عشر ايضاً ان يكون في وسط المربع مربع
من ضرب عشرة في عشرة ويحيط به اثنا عشر مربعاً كلّ واحد منها من
ضرب اربعة في اربعة تتفق جملة ايضاً. وتهيأ ان يكون في اربع زواياه
اربعة مربعات كلّ واحد منها من ضرب ثمانية في ثمانية وتتفق جملها
ايضاً. وكذلك في سائر هذه الاعداد.

560 وقد رُسم من المربعات التي يتهيأ القسمة فيها فيما بين الثمانية الى
العشرين وقُسم كلّ مربع منها على اكثر ما يتهيأ ان يتقسم عليه لتستدل
بذلك على ما يراد قسمته بعد هذه المربعات.

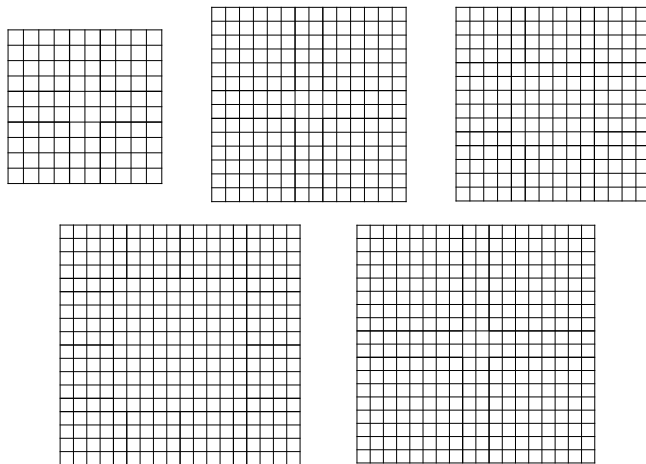
(47) فالما المربعات الفردية فانه يتعذر ان يُقسم مربع منها بمربعات
يكون عدد كلّ واحد منها فرداً ثمّ يودى الى هذه الشريطة لانّ المربع
الذى عدده فرد يحتاج الى واسطة واحدة فتمتّى جعل في المربع الكبير
565 عدّة مربعات عدد كلّ واحد منها فرد وأحتيج الى ان تتفق جملها أحتيج
الى ان يكون واسطة كلّ مربع منها مثل واسطة المربع الآخر ليعدّل بما
يبقى من الاعداد.

555 || A ١٨ [ثمانية عشر 555 || om. (pr.) مربع 555 || A (homoeotel.) om. ... واذا ... جملة
556 || D اثني عشر, A ١٢ [اثنا عشر 556 || A ١٠ في ١٠ [عشرة في عشرة 556 || A مربعاً [مربع
D ايضاً جملها, A حملتها ايضاً [جملة ايضاً 557 || A ٤ في ٤ [اربعة في اربعة 557 || D om. [منها
|| A om. [وقد رُسم ... هذه المربعات 560-2 || A (homoeotel.) om. [ويتهيأ ... ايضاً 557-9 ||
add. انسا الله [المربعات 562 || D.

(47) A: 21^r, 9-21^r, 20 (ed. ll. 934-944); D: 99^v, 11-99^v, 17.

tit. D : فصل

مربع 563 || D ولم نقف على أنّه قد ممكن ان تقسم مربع من المربعات [فالما ... بمربعات 563
D واحده [واسطة 565 || A عدته [عدده 565 || AD الواحد [كّل واحد 564 || A منها مربع [منها
566 || D om. [تتفق جملها احتيج الى ان 566-7 || D فرداً [فرد 566 || D om. [عدد 566 ||
567 || A الاخير [الآخر 567 || A om. [مثل 567 || A واحد [مربع 567 || A حملتها [جملها



Squares admitting of (such) a division, (of orders) from eight to twenty, have been constructed, each being divided in several possible ways; you will thereby rely on that for any desired division of other squares.¹⁶⁴

(§47. No division into odd-order subsquares)

As for the odd(-order) squares, it is impossible for such a square, being divided into squares with the order of each being odd, to satisfy this condition.¹⁶⁵ Indeed, a square of odd order necessarily has a single median number; so if one places in the main square a certain quantity of squares, each of odd order and supposed to display the same sum, the median number of any of these squares will have to be the same as that of any other in order that the equalization with the remaining numbers might be performed.¹⁶⁶

[Now of the numbers (to be placed) none may ever occur twice, and it may be put only in a single place; so if you put the median in all (these) squares and (then) omit their places altogether whilst putting consecutive

¹⁶⁴ See p. 124 *seqq.* (and *n.* 142). The filling of their parts will be explained later (§§ 48 *seqq.*). There is no 8×8 square in the extant text (see p. 125).

¹⁶⁵ This, incidentally, anticipates any reader's question about the division of the 18×18 and 30×30 squares into 36 squares of order 3 and 5, respectively. Generally, it is impossible for the whole square to be magic with its (equal) subsquares displaying equal sums (this being the 'condition').

¹⁶⁶ Since each subsquare must contain the same magic sum, their central cells should all contain half the sum of two complementary numbers. What follows is a few disorderly glosses attempting to complete the argument; namely, that either putting the same quantity in all central cells or leaving them all empty whilst filling in both cases the remainder of the squares with pairs of complements will in no way satisfy the magic condition. Argument reproduced in an early 11th-century text (*Un traité médiéval*, pp. 80 & 166, *ll.* 656–658).

[وليس في شيء من هذه الأعداد عدد مثني ولا يثبت إلا في موضع واحد فان اثبتت الواسطة في جميع المربعات واسقطت مكانها كلها واثبتت
570 أعدادًا على نسق [حتى تسقط مكان كل واسطتين] [عددين متقابلين] لم يكن هذا على شريطة الوفاق ولا عاد إلى ما بيننا في الأعداد الفردية (في المقالة) الأولى وتبطل الجملة الكبرى لتكرار بعضها واسقاط بعض].

(48) فاذا أردت أن ترسم الأعداد في مربع من المربعات التي قد قسمتها بمربعات لم يفصل بينها سطور لا تبقى بمربع تام فخذ واسطتي الأعداد
575 التي تقع في ذلك المربع فاثبتهما في مربع من المربعات التي قسمته عليها حيث كانت واسطتا مربعة اربعة في اربعة ثم كذلك على ذلك الرسم حتى تنتهي إلى آخر ذلك المربع. ثم خذ الطرفين اللذين تنتهي اليهما فاقمهما مقام الواسطتين واثبتهما في مربع آخر من تلك المربعات وافعل به مثل ما فعلت أولًا. وكذلك أبدًا حتى تأتي على ما تريد.

(49) وان قسمت مربعًا منها بمربعات يفصل بينها سطور لا تبقى بمربع

[اسقطت 570 || D ست A, شت [اثبتت 570 || D شني [عدد مثني 569 || A لتعبدل [ليعدل عددين 571 || D كما ست عدد A, كلها ثبتت عددًا [كلها واثبتت أعدادًا 570-1 || D اسقط فان العمل في ذلك *def. in D, hab. hic* [لم يكن ... بعض 571-3 || D عددان مقابلان [متقابلين A. ويعود [ولا عاد 572 || *v. enim cod. D, fol. 60^v - 61^v*] كما بيننا في الأعداد الفردية

(48) A: 21^r, 20-21^v, 7 (ed. ll. 945-952); D: 99^v, 18-99^v, 22.

tit. D : فصل

[فاثبتهما 576 || D يبقى [تفي 575 || A ثم فصل [لم يفصل 575 || D نرسم A, تقسم [ترسم 574 || A في 4 || 4 [اربعة في اربعة 577 || D مربع [مربعة 577 || A *in marg.* A [من 576 || AD فاثبتها [فاقمهما 578 || D ناخذ [خذ 578 || D يأتي على [تنتهي إلى آخر 578 || D *om.* [على ذلك 577 وكذلك ... ما 580 || D كما فعلته بالاول [مثل ما فعلت أولًا 579-80 || D فاقمتهما A, فاقمهما *om.* D. تريد

(49) A: 21^v, 7-21^v, 17 (ed. ll. 953-962); D: 99^v, 22-100^r, 4.

tit. D : فصل

581 || D بقى [تفي 581 || A سنها [بينها 581 || A منها مربع [مربعًا منها 581 || D فان [وان 581 583 || A الفضل [الفصل 583 || A مساوٍ [مشارك 582 || A فضل [يفصل 582 || D لمربع [بمربع

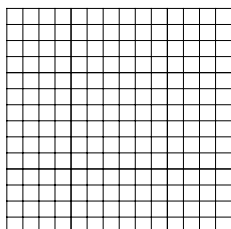
numbers [thus omitting the place of each pair of medians] [each pair of complements]¹⁶⁷, this will not meet the condition of magic square and ⟨not⟩ go back to what we have shown for (squares of) odd orders ⟨in⟩ the first ⟨chapter⟩, and the main sum will be wrong because of the repetition of some (numbers) and the omission of others.]

(§48. Filling a square divided into even-order subsquares)¹⁶⁸

If you wish to write the numbers in some square you have divided into squares not separated by strips insufficient to (contain) a complete square (proceed as follows). Take the two medians of the numbers allotted to this (main) square and put them in one of the squares into which you have divided the (main square), at the place of the two medians of the 4 by 4 square, and then (proceed) according to this (known way of) filling to the end of this square.¹⁶⁹ Take then the two terms you have attained, attribute them the rôle of medians and put them in some other square among these, and proceed as before.¹⁷⁰ (Do) always the same until you have finished with what you wanted.

(§49. Filling rectangular parts surrounding a central subsquare)

If (now) you have divided such a square into squares separated by strips insufficient to (contain) a complete square and these separations are in the sides without meeting the diagonal (proceed as follows).



α_4			α_1
β_4			β_1
γ_4			γ_1
	γ_3	γ_2	
	β_3	β_2	
	α_3	α_2	

— If the separation is 6 by 4, you will put in each (group of) four cells with (a group of) four (cells) below it eight complementary terms;¹⁷¹ with them, each row will be equalized with its conjugate [as we have explained

¹⁶⁷ *Obscurum per obscurius.*

¹⁶⁸ Thus the smallest subsquares, or inner squares within subsquares, are 4×4 .

¹⁶⁹ Taking thus $\frac{n^2}{2}$ and $\frac{n^2}{2} + 1$, then the decreasing sequence from the first and the increasing one from the second. For example, in the figure on p. 137, the first numbers put are 200 and 201, namely in the first subsquare (if, unlike the author, we consider putting the numbers from 1 on, we shall begin with the last subsquare).

¹⁷⁰ Since each subsquare is to display the same sum, they can be dealt with in any sequence.

¹⁷¹ Thus, in our above figure, each of the four α_i (or β_i, γ_i) with their complements lined up vertically.

تأم وكان ما يفصل من ذلك في الجوانب غير مشارك للقطر وكان ذلك
 الفصل ستة في أربعة فأتك تثبت في كل أربعة آيات تحتها أربعة ثمانية
 اطراف متقابلة فعدّل بها كل سطر بصاحبه [كما بيّنّا في صدر هذا
 الباب] وتفعل ذلك حتّى تأتى على الستّة. وان كان الفصل ستة في ثمانية
 عملت في الاربعة في الستّة الباقية مثل ذلك.

وليس يتهيأ ان يُقسم المربع اقسامًا يكون الفصل بينها فردًا لما بيّنّا
 وأما يكون الفصل أما ستة في أربعة وأما ستة في ثمانية او اثنين في
 أربعة او اثنين في ستة او اثنين في ثمانية وعلى هذا النحو ابدًا. واذا كان
 الفصل اثنين في أربعة وكان في الجوانب غير مشارك للقطر فالعمل في
 تعديله كما فسّرنا متقدّمًا.

(50) وان كان ذلك <الفصل> في الوسط حتّى يشارك القطر فيصير في
 وسط المربع يكون من ضرب اثنين في اثنين مثل ما يقع في العشرة
 في عشرة اذا جعل في زواياها أربعة مربّعات كل واحد منها من ضرب
 أربعة في أربعة ومثل ما يقع في مربع أربعة عشر اذا جعل في اربع
 زواياه أربعة مربّعات كل واحد منها من ضرب ستة في ستة.

583 AD || اسطر [آيات 583 || A 4] أربعة 583 || A 6 في 4 [ستّة في أربعة
 584-5 || A 584 صاحب [بصاحبه 584 || D يعدّل [فعدّل 584 || A 8 ثمانية 583 || A 4] (ult.) أربعة
 الفصل 585 || D ويفعل, A فعل [وتفعل 585 || D (v. enim D, fol. 33^v) الكتاب [هذا الباب
 الاربعة الباقية وفي [الاربعة في الستّة الباقية 586 || A 6 في 8 [ستّة في ثمانية 585 || D الفعل
 587 || D فه, A فيها [بينها 587 || A الفضل [الفصل 587 || D الاربعة الباقية في الستّة, A الستّة
 || D او [وأما 588 || A 6 في 4 [ستّة في أربعة 588 || AD الفضل [الفصل 588 || D بيّنّه [بيّنّا
 A 2 في 4 [اثنين في أربعة 588-9 || A وأما corr. ex 588 || A 6 في 8 [ستّة في ثمانية 588
 589 || D (sic) 1 [هذا 589 || A 2 في 8 [اثنين في ثمانية 589 || A 2 في 6 [اثنين في ستة 589
 [فالععمل 590 || A 2 في 4 [اثنين في أربعة 590 || AD الفضل [الفصل 590 || D فادا [واذا
 العمل 591 || D om. D. كما 591 || D والعمل

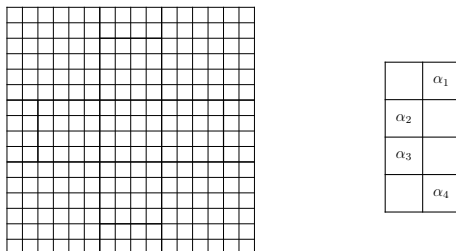
(50) A: 21^v, 17-22^r, 8 (ed. ll. 963-973); D: 100^r, 4-100^r, 11.

في 10 [في العشرة في عشرة 4-593 || A 2 في 2 [اثنين في اثنين 593 || AD وبصير [فيصير 592
 4 في [أربعة في أربعة 595 || D om. منها 594 || A اربع [أربعة 594 || D في العشرة, A في 10
 اربع [أربعة 596 || D om. اربع 595 || A 14 [أربعة عشر 595 || D مربع [مربع 595 || A 4
 597 || A الوسطين [الواسطتين 597 || A 6 في 6 [ستّة في ستة 596 || D om. منها 596 || A

at the beginning of this section¹⁷²]. You proceed like that until you have completed the six (rows).

— If the separation is 6 by 8, you will treat the remaining 4 by 6 in the same way.

It is not possible that the square be divided into parts with the separation between them being odd (in dimensions) because of what we have explained;¹⁷³ as a matter of fact, the separation will be 6 by 4, or 6 by 8, or 2 by 4, or 2 by 6, or 2 by 8, and so on always. If the separation is 2 by 4 and is situated in the sides without meeting the diagonal, the treatment for its equalization will be as we have explained previously.¹⁷⁴



(§ 50. Filling the central part of a cross placed in the middle)

If this separation is in the middle and thus meets the diagonal, there appears in the centre of the (main) square a square with size 2 by 2. This for instance occurs in the (square of size) 10 by 10 when are placed in its corners four squares each of size 4 by 4, or in the square of fourteen when are placed in its four corners four squares each of size 6 by 6.¹⁷⁵

¹⁷² In § 42, or else in § 20. (See § 31, *n.* 97, similar interpolated reference.)

¹⁷³ The case of odd-order subsquares having been ruled out (§ 47), the sides of the strips are necessarily even. If not (case of strips 1×4 , generally $1 \times 4m$), the two opposite strips will be associated (see examples in the subsequent note).

¹⁷⁴ Using a neutral placing (§ 42), see our figure. The separations 6×4 and 6×8 (generally, $6 \times 4m$) have just been treated, leaving the cases $2 \times 2m$. A pair of opposite rectangles 2×6 may be reduced to the case 6×4 by combining them. A strip 2×8 , or $2 \times 4m$, is reducible to two, or m , rectangles 2×4 . Note also that the separations $2m \times 4$ and $2m \times 6$, m odd, can be reduced to rectangles 2×4 by removing 4×4 squares. Thus we find among the examples of larger squares: a 6×4 rectangle reduced to one 4×4 square and either a 2×4 rectangle (pp. 141, 153) or two disjunctive 1×4 strips (p. 139); a 6×8 rectangle reduced to filling either two 4×4 squares and two 2×4 rectangles (p. 153) or one 6×6 square and two disjunctive 1×6 strips, each associated with its opposite horizontally and vertically (p. 151); a 10×4 rectangle reduced to two 4×4 squares and two disjunctive 1×4 strips (p. 145). The examples with a central cross (see below) also lead to filling 2×4 rectangles.

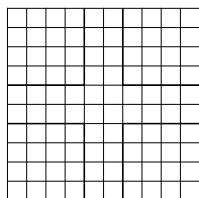
¹⁷⁵ See pp. 155 & 157. These examples have been mentioned in § 46, together with the example of an 18×18 square. What follows is generally applicable to evenly-odd squares.

600

605

(51) $\mathcal{A}$: 22^r,8-22^r,11 (ed. ll. 974-977); $\mathcal{D}$: 100^r,11-100^r,13.

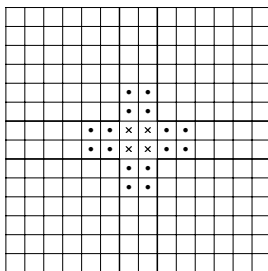
ستينين [بيتين 604 || $\mathcal{A}$ ٢ في ٦] اثنين في ستة 603 || $\mathcal{D}$ المربع [التصليب 603 || $\mathcal{A}$ الدي [ما 603
اثنان في اربعة 605-6 || $\mathcal{A}$ جهات [الجهات 605 || $\mathcal{A}$ ٦ في ٢] ستة في اثنين 605 || $\mathcal{D}$ بيتين $\mathcal{A}$,
 $\mathcal{A}$. ٢ في ٤.



	+3	-3	
	↑	↑	
	$\frac{n^2}{2} + 3$	$\frac{n^2}{2}$	$\Rightarrow +2$
	$\frac{n^2}{2} + 1$	$\frac{n^2}{2} - 2$	$\Rightarrow -2$

(You will proceed in this case) as follows. You write the two medians of the whole set of numbers¹⁷⁶ in two diagonally opposite corners of the square with size 2 by 2 in the centre of the main square. Omitting two numbers after the two medians,¹⁷⁷ you take the next two and place them in the two opposite corners left.¹⁷⁸ At this point, you will find that the row containing the two small numbers has a deficit of 3 relative to its sum due, that the row parallel to it has an excess of 3, that the row containing the larger of the two small (numbers) and the larger of the two large (numbers) has an excess of 2, and that (the row) parallel to it has a deficit of 2.

(§51. Case of order $n = 14$)



If then what remains in the cross is, in its four directions,¹⁷⁹ 2 by 6,¹⁸⁰ you will have to equalize the (central) square by means of two cells on each side,¹⁸¹ in this way, the central (then filled part of the) cross will be (of size) 6 by 2 and there will remain, to complete the cross on the

¹⁷⁶ Thus the two medians of the main square ($\frac{n^2}{2}$ and $\frac{n^2}{2} + 1$), which are complementary numbers.

¹⁷⁷ One after each. 'Omitting': *taraka*, perhaps translating ὑπερβαίνειν.

¹⁷⁸ In our figure, with the larger one above.

¹⁷⁹ On each side of the 2×2 central square.

¹⁸⁰ Case of the 14×14 square, see figure.

¹⁸¹ That is (considering also the complements) by means of two pairs of cells on each side of the 2×2 central square.

(52) والعمل في ذلك < أنك > اذا اثبتت الاربعة الاعداد التي وصفنا في الوسط نظرت الى السطر الذي ينقص ثلثة فاثبت في البيتين اللذين يتصلان به من احدى الجهتين اقل العددين اللذين تركتهما بعد الواسطتين والعدد الكثير الذي بعد العددين < الكثيرين > اللذين اثبتتهما في الوسط 610 واثبت في البيتين اللذين يتصلان به من الجهة الاخرى العدد الاقل من العددين < القليلين > اللذين يتلوان < جميع > ذلك والعدد الاكثر من العددين < الكثيرين > اللذين يتلوان جميع ذلك واثبت بازاء كل واحد منها ما يقابله. فاذا فعلت ذلك فقد عدلت هذه الستة الايات.

ثم تنظر الى السطر الذي ينقص اثنين فاثبت في احدى الجهتين < في 615 البيتين > اللذين يتصلان به من الاطراف التي انتهيت اليها اقلها ثم

(52) A: 22^r, 11 - 22^v, 4 (ed. II. 978-992); D: 100^r, 13 - 100^r, 21.

D فاست [فاثبت 608 || A ٣ [ثلثة 608 || D فنطرت [نظرت 608 || AD است [اثبتت 607
استهما [اثبتتهما 610 || A فصداً [بعد 610 || D الكبير [الكثير 610 || A الواسطين [الواسطتين 609
AD القليل [الاقل 611 || A يصلا [يتصلان 611 || D الزاويتن, A الزاويه [الوسط 610 || AD
منهما [منها 614 || A om. [جميع 613 || D الكبير, A الكبير [الاكثر 612 || A العدد [والعدد 612
617 || A سصل [يتصلان 616 || AD (الجهتين sc.) اللتين [اللذين 616 || A ٢ [اثنين 615 || AD

four sides, (strips of size) 2 by 4 which you will equalize in the manner explained.¹⁸²

(§ 52. Equalizing the cross)

(In order to equalize the central square) you proceed as follows.¹⁸³ After putting in the centre the four numbers which we have indicated, you consider the row with a deficit of 3.¹⁸⁴ Put, in the pair of cells adjacent to the (central square), on one of the two sides, the lesser of the two numbers you have omitted after the two medians (thus $\frac{n^2}{2} - 1$) and the large number following the two large numbers you have written in the centre (thus $\frac{n^2}{2} + 4$);¹⁸⁵ and put, in the two cells adjacent to the (central square) on the other side (of this same vertical row), the lesser of the two small numbers following all that (thus $\frac{n^2}{2} - 4$) and the larger of the two large numbers following all that (thus $\frac{n^2}{2} + 6$); and put opposite to each its complement. Once you have done that, you will have equalized these six (pairs of) cells.¹⁸⁶

				†			
			$\frac{n^2}{2} - 3$	$\frac{n^2}{2} + 4$			
			$\frac{n^2}{2} + 2$	$\frac{n^2}{2} - 1$			
	$\frac{n^2}{2} - 9$	$\frac{n^2}{2} + 9$	$\frac{n^2}{2} + 3$	$\frac{n^2}{2}$	$\frac{n^2}{2} + 7$	$\frac{n^2}{2} - 7$	
	$\frac{n^2}{2} + 10$	$\frac{n^2}{2} - 8$	$\frac{n^2}{2} + 1$	$\frac{n^2}{2} - 2$	$\frac{n^2}{2} - 6$	$\frac{n^2}{2} + 8$	★
			$\frac{n^2}{2} + 5$	$\frac{n^2}{2} - 4$			
			$\frac{n^2}{2} - 5$	$\frac{n^2}{2} + 6$			

Next, you examine the row in deficit of 2.¹⁸⁷ Put on one of the two sides, in the two cells adjacent to the (central square), the lowest of the terms you have reached ($\frac{n^2}{2} - 6$), then the larger of the two large

¹⁸² Using neutral placings, as seen in §§ 42–43. But, as said, we are first to eliminate the differences displayed by the central square.

¹⁸³ As told in the previous section, we must now equalize the central square, already filled (§ 50), by means of two pairs of cells in each branch. Our addition, in brackets, of the numbers alluded to (and of the figure) should make the text less abstruse.

¹⁸⁴ Marked with † in our figure.

¹⁸⁵ Thus, the smallest number placed is now $\frac{n^2}{2} - 2$ and the largest, $\frac{n^2}{2} + 4$.

¹⁸⁶ ‘these six cells’: we may consider the equalization on just one row since *ipso facto* the complements will equalize its conjugate.

¹⁸⁷ Marked with ★ in our figure. With the complements just placed, the next available numbers are $\frac{n^2}{2} - 6$ and $\frac{n^2}{2} + 7$.

«اكثر» الكثيرين اللذين تنتهى اليهما وكذلك فى الجهة الاخرى واثبت
بازاء كل عدد ما يقابله. فاذا فعلت ذلك فقد عدلت هذه الستة الايات.
فتعدل حينئذ ما يبقى من الجهات وهو اربعة فى اثنين فى كل جهة بما
فسرت لك.

620

(53) وان كان ما يبقى بعد المربع الاوسط من كل جهة اثنين فى اربعة
جعلت ما يعدل به السطر الذى ينقص ثلاثة فى اربعة ايات من جهة
واحدة من جهتيه وما يعدل به السطر الذى ينقص اثنين فى اربعة ايات
«من جهة واحدة» من جهتيه ولم تحتج الى ان تجعل ذلك تكييفاً له من
الاربع الجهات. فاذا فعلت ذلك عدلت كل اربعة ايات يبقى فى الجهتين
الباقيتين بما فسرت لك.

625

(54) وكذلك تعمل اذا بقى اثنان فى ثمانية او اثنان فى عشرة. فانك تعدل
الاوسط واذا بقى ثمانية «عدلت الوسط» فى جهتين فقط باربعة ايات
اربعة ايات واذا بقى عشرة عدلت من الاربع الجهات ببيتين بيتين.

om. D || [الايات 618 *D* تعدلت] عدلت 618 *A* || هذا [ذلك 618 *D* الكبير من] الكثيرين
[بما فسرت لك 619-20 *A* || ٤ فى ٢ [اربعة فى اثنين 619 *A* وهى] وهو 619 *om. D* || من 619
D. كما فسرت.

(53) *A*: 22^v, 4-22^v, 9 (ed. ll. 993-998); *D*: 100^r, 21-100^r, 24.

622 *A* || ٣ فى ٤ [ثلاثة فى اربعة 622 *A* || ٢ فى ٤ [اثنين فى اربعة 621 *A* فان] وان 621
om. (homoeotel.) A || [وما يعدل ... من جهتيه 623-4 *A* حمله] جهة 622 *A* ايانا [ايات
626 *om. D* || ايات 625 *A* || ٤ [اربعة 625 *AD* مكتنفاً] تكييفاً 624 *om. D* || الى 624
D (v. enim *D*, fol. 33^v). فسرنا [فسرت]

(54) *A*: 22^v, 9-22^v, 17 (ed. ll. 999-1006); *D*: 100^r, 24-100^v, 3.

[اثنان فى ثمانية 627 *A* فى باقى] اذا بقى 627 *D* تعمل [تعمل 627 *A* فكدلك] وكذلك 627
628 *A* || ٨ [ثمانية 628 *AD* اذا] واذا 628 *A* || ٢ فى ١٠ [اثنان فى عشرة 627 *A* || ٢ فى ٨
A ٤ اسات ٤ اسات [باربعة ايات اربعة ايات 628-9 *D* من جهتين] فى الجهتين
بيتين بيتينه [ببيتين بيتين 629 *D* عدلته] عدلت 629 *A* || ١٠ [عشرة 629 *A* وان] واذا 629

(numbers) you have reached $(\frac{n^2}{2} + 8)$; and (proceed) in like manner (as in the previous case) for the other side.¹⁸⁸ Put opposite to each number its complement. Once you have done this, you will have equalized these six (pairs of) cells.

At that point, you will equalize the remainder on the (four) sides, (of size) 4 by 2 on each side, using what I have explained to you.¹⁸⁹

(§ 53. Case of order $n = 10$)¹⁹⁰

(But) if what is left (in the cross), excepting the central square, is on each side 2 by 4, you will place that which will equalize the row in deficit of 3 in four cells on (just) one of the two sides of the (central square), and that which will equalize the row in deficit of 2 in four cells on (just) (one of) its two sides: you do not need to do this with sharing out on its four sides. Once you have done that, you will equalize each remaining (group of) four cells in the two other sides using what I have explained.¹⁹¹

(§ 54. Extension to higher orders)

You will proceed likewise for remainders of 2 by 8 or 2 by 10.¹⁹² You will (first) equalize the centre; (namely,) when there remain eight (cells), you equalize the centre on two sides only, using two groups of four cells, and when there remain ten (cells), you do it on all four sides, using pairs of cells. (Then) you will equalize what remains, (consisting of strips of

¹⁸⁸ As for the other side of the vertical branch, see above. We shall thus place the lower of the next small pair $(\frac{n^2}{2} - 7, \frac{n^2}{2} - 8)$, thus $\frac{n^2}{2} - 8$, and the higher of the next large pair $(\frac{n^2}{2} + 9, \frac{n^2}{2} + 10)$.

¹⁸⁹ Each remaining part of the cross receiving one neutral placing. See example, p. 157 ($n = 14$).

¹⁹⁰ With, on each side of the central (2×2) square, a 2×4 rectangle; the equalization of the central square *may* then be performed on two sides only, see our figure (this is by no means necessary).

¹⁹¹ Example on p. 155.

¹⁹² Disregarding the 2×2 central square, thus case of, respectively, order 18 (with equalization of the central square on two of its sides only) and 22 (distributing the equalization among its four sides). Once again, this distinction is superfluous since a neutral placing does not need to occupy consecutive cells.

630

وتعدّل ما يبقى وهو اثنان في اربعة على ما فسّرتُ لك متقدّمًا.

واذا نظرتُ الى المربّعات التي رسمتها لك ووقفتَ على مواضع الاعداد وترتيبها استدلتُ بذلك على ما فسّرتُ لك.

D (v. *fr*) فرياه [فسّرتُ لك 630 *D* اربعة اربعة (*sic*) *A*, ٢ في ٤] اثنان في اربعة 630 *AD* || 631 *D* في [الى 631 *A* (*corr.* ذلك) وذلك أنّك اذا] واذا 631 *D*, fol. 33^v) || 632 *post haec add. A*: ممعونة الله: [فسّرتُ لك 632 *A* ووقعت] ووقفتَ 631 *A* استها [رسمتها لك ان شا الله والله *post quod add.*: فسرت *D hab.*; وحسن توفيقه فاعتبره تصلب ان شا الله تعالى اعلم.

*Post tit. prae*b. *A* (fol. 23^r - 27^r) *figuras quadratorum*. وهذه امثلة ما ذكرنا في الافراد والازواج جميعًا

size) 2 by 4, as I have explained to you before.¹⁹³

After examining the squares which I have constructed and being acquainted with how to place the numbers and arrange them, you will (be able) to rely on it for what I have explained to you.¹⁹⁴

¹⁹³ Two examples, see pp. 159 & 161 ($n = 18$).

¹⁹⁴ Rather (and against the MSS.): ‘what I have *not* explained to you’. That is, if this specifically refers to the method of the cross, extension of it to orders $n = 8t + 2$ and $n = 8t + 6$ with $t \geq 3$.

Excerpta e cod. D, fol. 95^v-96^v.

القسم الثاني في المزاج المعتدل وهو للمفضل بن ثابت واورده ابو الوفاء البوزجاني في *Inc. (D, 95^v)* كتابه. هذا القسم اخت الموشع لاشتراكهما في صور المسطّات ألا ان هذا اعمّ منه لان الموشع لا ينتج منه شكل اثنين وهذا ينتج. والموشع يأتي مسطّه فردًا وزوجًا وهاهنا لا يأتي إلا زوجًا وان حمل الموشع يكون مختلفة التفاضل وهاهنا كلّها يكون متفقة. (...)

الباب الثاني في اثبات اساس زوج لمرّبعات كثيرة في لوح كبير اذا اردنا اثبات مرّبعات كثيرة متساوية الاشكال مختلفة الاعداد السّيارة فيها من الصغار والكبار ألا أنّها متوافقة الجمل طولًا وعرضًا وقطرًا وهذا هو الذي يسمّى له المزاج المعتدل رجعنا الى المقالات المتقدمة واخترنا فيها لوحًا زوجًا كثر الوفق جعلناه اساسًا لمرّبعات كثيرة وحفظناه وعرفنا فيه الاعداد الصغار الزائدة مبادئها ومقاطعها وكذلك الاعداد الكبار الناقصة مبادئها للحاجة اليها كما اوردنا هاهنا مثالًا للاربعة واثبتنا الاعداد الصغار فيها بالحروف من أ الى ح والكبار بالحساب من ٩ الى ١٦ هكذا.

د	يه	ى	ه
يد	ا	ح	يا
ز	يب	يج	ب
ط	و	ج	يو

اللوحة الاساسية

فاذا اردنا معرفة مسطّ اثنين في اثنين وهو اربعة مسطّات كلّ واحد منها اربعة في اربعة رجعنا الى لوح ثمانية وعرفنا مقاطع اعداده الصغار للمربّعات الاربعة وهي ح يو كد لب واعداده الكبار وهي م مح نو سد ومبادئ الصغار أ ط يز كه ولكبار ل ج ما مط نز اثبتنا في المسطّ الاول الصغار من أ الى ح والكبار من نز الى سد على ترتيب الاساس وفي الثاني الصغار من ط الى يو والكبار من مط الى نو وفي الثالث الصغار من يز الى كد والكبار من ما الى مح وفي الرابع الصغار من كه الى لب والكبار من ل ج الى م.

كح	لط	لد	كط	يب	نه	ن	يج
لح	كه	لب	له	ند	ط	يو	نا
لا	لو	لز	كو	يه	نب	نج	ى
لج	ل	كز	م	مط	يد	يا	نو
د	سج	نح	ه	ك	مز	مب	كا
سب	ا	ح	نط	مو	يز	كد	مج
ز	س	سا	ب	كج	مد	مه	بح
نز	و	ج	سد	ما	كب	يط	مح

لوح ثمانية جملة كلّ صفّ ١٦٠. وقد قسم
اربعة مسطّات وجمل صفوفها متساوية من
جميع الجهات وجملة كلّ صفّ منها ١٣٠

Division into equal subsquares

There are six examples of composite squares with equal subsquares all displaying the same magic sum. Their filling always follows the same pattern, namely that of the bordered squares of even orders already seen; we have added each time, as a reminder, the arrangement of the small numbers in the borders (except for the 4×4). Since subsquares displaying same sums can be filled in any order, we have added a table showing this order for the transcription (remember that the squares are filled with the descending sequence of smaller numbers, thus the first subsquare to be filled is that containing $\frac{n^2}{2}$).

In MS. $\mathcal{D}$, all these squares are given a heading, later on also a short ‘notice’. Their characteristics are pointed out, together, at times, with further remarks. Here the Arabic text will be given below the figures of the squares.

In his introduction to this part, the author of the compilation of MS. $\mathcal{D}$ tells us that he finds this arrangement ‘more general’ than the arrangement with subsquares displaying unequal magic sums, for, in this case, a 2×2 arrangement of subsquares is possible whereas it would be impossible with subsquares displaying unequal sums. On the other hand, he adds, the other type of arrangement admits subsquares of odd orders, with sums in arithmetical progression (see Commentary, below p. 213). He also notes the main characteristic of the present type of arrangement: the configuration of the basic subsquare (*asās*) will be repeated in each subsquare, half of which will be filled with the continuous sequence of small numbers and the other half with the continuous sequence of their complements. As an example, he presents the case of the 4×4 square producing the 8×8 composite square.

4	15	10	5
14	1	8	11
7	12	13	2
9	6	3	16

28	39	34	29	12	55	50	13
38	25	32	35	54	9	16	51
31	36	37	26	15	52	53	10
33	30	27	40	49	14	11	56
4	63	58	5	20	47	42	21
62	1	8	59	46	17	24	43
7	60	61	2	23	44	45	18
57	6	3	64	41	22	19	48

كج	فكو	ك	قكد	قيح	كد	نط	ص	نو	فخ	فب	س
قيز	لب	قيه	قي	لج	كح	فا	سح	عط	عد	سط	سد
كو	قيد	كط	لو	قيا	قيط	سب	سح	سه	عب	عه	فخ
كه	له	قيب	قيج	ل	قك	سا	عا	عو	عز	سو	فد
فكج	قط	لد	لا	قيو	كب	فز	عج	ع	سز	ف	نح
فكا	يط	فكه	كا	كز	فكب	فه	نه	فط	نز	سج	فو
ما	قح	لح	قو	ق	مب	ه	قمد	ب	قنب	قلو	و
صط	ن	صز	صب	نا	مو	قله	يد	قلج	فكح	يه	ي
مد	صو	مز	ند	صيح	قا	ح	قلب	يا	يح	فكط	قلز
مخ	نخ	صد	صه	مح	قب	ز	يز	قل	قلا	يب	قلح
قه	صا	نب	مط	صيح	م	قما	قكرز	يو	يح	قلد	د
فخ	لز	قر	لط	مه	قد	قلط	ا	قمج	ج	ط	قم

(12×12) $\mathcal{A}$: 26^v (ed. p. 314); $\mathcal{D}$: 96^v .

tit. $\mathcal{A}$: ١٢ في ١٢

لوحي اثني عشر جمل صفوفه متساوية وجملة كلّ صفّ منها ٨٧٠. وقد قسم باربعة مسمّطات محلّقة : $\mathcal{D}$.
كلّ مسمّط ستّة في ستّة وجمليها من جميع جهاتها مساوية وجملة كلّ صفّ منها ٤٣٥.

A latere quadrati in $\mathcal{A}$ inven. ٢ (sc. 2 quadrata).

$quasi$ $(6, 3)$ $\mathcal{A}$ || ز || نا $(4, 4)$ || يا || $\mathcal{A}$ || لح $(10, 6)$ || و $\mathcal{A}$ || $corr. ex$ $(2, 8)$ || ف || $\mathcal{A}$ || مط $(2, 11)$ || سط
 $(5, 1)$ || ا || $\mathcal{D}$ (id.) ج || $(4, 1)$ || قمج || $\mathcal{D}$ (v. præced. loculum) ط $(3, 1)$ || ج || $\mathcal{A}$ || ملر $(5, 2)$ || فكرز || $\mathcal{A}$ || ن
قمج (id.) $\mathcal{D}$.

23	126	20	124	118	24	59	90	56	88	82	60
117	32	115	110	33	28	81	68	79	74	69	64
26	114	29	36	111	119	62	78	65	72	75	83
25	35	112	113	30	120	61	71	76	77	66	84
123	109	34	31	116	22	87	73	70	67	80	58
121	19	125	21	27	122	85	55	89	57	63	86
41	108	38	106	100	42	5	144	2	142	136	6
99	50	97	92	51	46	135	14	133	128	15	10
44	96	47	54	93	101	8	132	11	18	129	137
43	53	94	95	48	102	7	17	130	131	12	138
105	91	52	49	98	40	141	127	16	13	134	4
103	37	107	39	45	104	139	1	143	3	9	140

		3							1		
		2							4		

<i>v</i>	·	<i>ii</i>	·	·	<i>vi</i>
·	<i>iv</i>	·	·	<i>v</i>	<i>x</i>
<i>viii</i>	·	<i>i</i>	<i>viii</i>	·	·
<i>vii</i>	<i>vii</i>	·	·	<i>ii</i>	·
·	·	<i>vi</i>	<i>iii</i>	·	<i>iv</i>
·	<i>i</i>	·	<i>iii</i>	<i>ix</i>	·

Commentary in D: Square of order $n = 12$, with four 6×6 subsquares (*musammatāt*). (Here and in what follows, the various magic sums are indicated.)

سط	عد	عط	سح	كط	قيد	قيط	كح	سا	فب	فز	س
عه	عب	سه	عح	قيه	لب	كه	قيح	فخ	سد	نز	فو
سو	عز	عو	عا	كو	قيز	قيو	لا	نخ	فه	فد	سح
ف	سز	ع	عج	قك	كز	ل	قيج	فخ	نط	سب	فا
كا	فكب	كز	ك	نح	ص	صه	نب	يخ	قل	يله	يب
كج	كد	يز	ككو	صا	نو	مط	صد	قلا	يو	ط	قلد
يح	فكه	فكد	كج	ن	صيح	صب	نه	ي	قلج	يله	يه
كح	يط	كب	فكا	صو	نا	ند	فط	قلو	يا	يد	فكط
مه	صح	فخ	مد	ه	قلح	قج	د	لز	قو	قيا	لو
صط	مح	ما	قب	قلط	ح	ا	قرب	قر	م	لج	قى
مب	قا	ق	مز	ب	فها	قم	ز	لد	قط	قخ	لط
قد	مح	مو	صز	قمد	ج	و	قلز	قيب	له	لح	قه

(12 × 12) $\mathcal{A}$: 27^r (ed. p. 315); $\mathcal{D}$, 96^v.

tit. $\mathcal{A}$: ١٢ في ١٢

tit. $\mathcal{D}$: لوح اثني عشر متساوي الجمل كل صف ٨٧٠. وقد قسم بتسعة مستطوات كل مستط أربعة في أربعة وجمل صفوف كل مستط متساوية من جميع الجهات وجملة كل صف منه ٢٩٠.

In ima pag. $\mathcal{A}$ rec. manu: 12 × 12 = 144.

صز [(6, 8) ص || $\mathcal{A}$ فك [(5, 9) قك || $\mathcal{A}$ (iter. in marg.) و corr. ex [(1, 9) ف || $\mathcal{A}$ ب [(10, 12) فب
corr. [(9, 4) لز || $\mathcal{A}$ فد [(4, 4) مد || $\mathcal{A}$ قط [(8, 5) فط || $\mathcal{A}$ فه [(7, 5) ند || $\mathcal{A}$ فكح [(1, 7) قكج || $\mathcal{A}$
 $\mathcal{A}$ ممب [(9, 1) قيب || $\mathcal{A}$ ممل [(5, 1) قمد || $\mathcal{A}$ مج [(2, 1) مد || $\mathcal{A}$ لد ex

60	87	82	61	28	119	114	29	68	79	74	69
86	57	64	83	118	25	32	115	78	65	72	75
63	84	85	58	31	116	117	26	71	76	77	66
81	62	59	88	113	30	27	120	73	70	67	80
12	135	130	13	52	95	90	53	20	127	122	21
134	9	16	131	94	49	56	91	126	17	24	123
15	132	133	10	55	92	93	50	23	124	125	18
129	14	11	136	89	54	51	96	121	22	19	128
36	111	106	37	4	143	138	5	44	103	98	45
110	33	40	107	142	1	8	139	102	41	48	99
39	108	109	34	7	140	141	2	47	100	101	42
105	38	35	112	137	6	3	144	97	46	43	104

A 10x10 grid with numbers 1 through 9 placed in various cells. The numbers are: 2 at (1,3), 6 at (1,5), 1 at (1,9), 8 at (4,2), 3 at (4,5), 7 at (4,9), 5 at (8,2), 9 at (8,5), and 4 at (8,9).

Remark. Every other subsquare is filled in turn; same below, pp. 135 & 137. This example is mentioned by Būzjānī (together with the 8×8 square seen above, pp. 124–125), see edition of his text, pp. 185–186 and 241, or *Magic squares in the tenth century*, B.26.

فخ	فی	قط	قشط	قن	قنا	قنب	قد	لط	مو	مه	ریج	رید	ریه	ریو	م
قب	قیه	قمو	قیب	قمد	قلح	قیو	قنه	لح	نا	ری	مح	رح	رب	نپ	ریط
قنو	قلز	قکد	قله	قل	قکه	قک	قا	رک	را	س	قسط	قصد	سا	نو	لز
ق	قیح	قلد	قکا	قکح	قلا	قلط	قنز	لو	ند	قصح	نز	سد	قصه	رج	رکا
قنح	قیز	قکرز	قلب	قلج	قکب	قم	صط	رکب	نیج	نیج	قصو	قصرز	نیج	رد	له
قنط	قمج	قکط	قکو	قکج	قلو	قید	صح	رکج	رز	قصج	سب	نط	ر	ن	لد
صز	قما	قیا	قمه	قیج	قیط	قنب	قس	لج	ره	مز	رط	مط	نه	رو	رکد
قنج	قنز	قمح	قخ	قز	قو	قه	قند	ریز	ریا	ریب	مد	مج	مب	ما	ریخ
ز	ید	یج	رमे	رمو	رمز	رمح	ح	عا	عح	عز	قفا	قنب	قنج	قند	عب
و	یط	رمب	یو	رم	رلد	ک	رنا	ع	خج	قحح	ف	قعو	قع	فد	قفز
نب	رلج	کح	رلا	رکو	کط	کد	ه	قنح	قسط	صب	قنر	قنب	صیح	خج	سط
د	کب	رل	که	لپ	رکز	رله	ریج	صح	فو	قسو	فط	صو	قنح	قعا	قفط
رند	کا	لا	رکج	رکط	کو	رلو	ج	قص	فه	صه	قسد	قسه	ص	قعب	سز
رنه	رلط	رکه	ل	کز	رلب	یج	ب	قصا	قعه	قسا	صد	صا	قنح	فب	سو
ا	رلز	یه	رما	یز	کج	رلج	رنو	سه	قنح	عط	قعرز	فا	فز	قعد	قصب
مط	رمج	رمد	یب	یا	ی	ط	رن	قغه	قعط	قف	عو	عه	عد	عج	قنو

لوح ستة عشر في ستة عشر متساوي الجمل من جميع جهاته وكل صف من صفوفه ٢٠٥٦. وقد قسم *tit.*:
 باربعة مربعات مسمطة محلاة كل مسمط منها ثمانية في ثمانية وكل صف من كل مسمط منه ١٠٢٨.

(5, 5) فط || *ut vid.* قعح (8, 6) قفح || قعح (3, 7) قع || قيج (15, 11) قيج || قنو *quasi* (9, 13) قنز
 || *ut vid.* نط

نب	رز	رب	نح	صب	قصر	قصب	صبح	س	قصط	قصد	سا	تكد	قله	قل	فكه
رو	مط	نو	رج	قسو	فط	صو	قصح	نر	سز	سد	قصه	قلد	فكا	فكح	قلا
نه	رد	ره	ن	صه	قصد	قسه	ص	نح	قصو	قصر	نح	فكر	قلب	فلج	فكب
را	ند	نا	رح	قسا	صد	صا	قصح	سب	نط	ر	فكط	فكو	فكج	قلو	
عو	فنج	فمح	عز	كح	رلا	ركو	كط	قح	قنا	قمو	قط	مد	ريه	رى	مه
ققب	عج	ف	فقط	رل	كه	لب	ركز	قن	قه	قيب	قمز	ريد	ما	مح	ريا
عط	قف	قفا	عد	لا	ركج	ركط	كو	قيا	قمح	ققط	قو	مز	ربب	ريج	مب
قعر	عح	عه	قفد	رکه	ل	كز	رلب	قمه	قى	قز	قنب	رط	مو	مج	ريو
ك	رل	رلد	كا	فد	قعه	قع	فه	لو	ركج	ريج	لز	قيو	قمج	قلح	قيز
رلح	يز	كد	رله	قعد	فا	فح	قعا	ركب	لج	م	رلط	قنب	قيج	قك	قلط
كج	رلو	راز	نح	فز	قعب	قعب	فب	لط	رك	ركا	لد	قبط	قم	قما	قيد
رلج	كب	يط	رم	قسط	فو	فح	قعو	ريز	لح	له	ركد	قلز	قيح	قيه	قمد
نح	قصا	قفو	سط	د	رنه	رن	ه	ق	قنط	قند	قا	يب	رمز	رمب	نح
قص	سه	عب	قفز	رند	ا	ح	رنا	قنح	صز	قد	قنه	رمو	ط	يو	رمح
عا	قفح	قفط	سو	ز	رنب	رنج	ب	قح	قنو	قز	صح	يه	رمد	رमे	ى
قنه	ع	سز	قصب	رمط	و	ج	رنو	قنج	قب	صط	قس	رما	يد	يا	رمح

(16 × 16) D : 97^v.

لوح ستّة عشر في ستّة عشر وجملة كلّ صفّ من صفوفه ٢٠٥٦. وقد قُسم بستّة عشر مسمّطًا كلّ *tit.*: مسمّط منها اربعة في اربعة وجملة صفّ كلّ مسمّط ٥١٤. وهو ايضًا مقسوم باربعة مربعات كلّ مربع من ضرب ثمانية في ثمانية وكلّ صفّ من صفوف كلّ واحد منها ١٠٢٨. وكذلك كلّ ثلاثة مسمّطات في ثلاثة مسمّطات اعنى اثني عشر في اثني عشر متوافقة وكلّ (صفّ من) صفوفها ١٥٤٢.

قفز *quasi* [(9,5) قعو || (قن *forsan ex* corr. e)] [(7,11) قه

52	207	202	53	92	167	162	93	60	199	194	61	124	135	130	125
206	49	56	203	166	89	96	163	198	57	64	195	134	121	128	131
55	204	205	50	95	164	165	90	63	196	197	58	127	132	133	122
201	54	51	208	161	94	91	168	193	62	59	200	129	126	123	136
76	183	178	77	28	231	226	29	108	151	146	109	44	215	210	45
182	73	80	179	230	25	32	227	150	105	112	147	214	41	48	211
79	180	181	74	31	228	229	26	111	148	149	106	47	212	213	42
177	78	75	184	225	30	27	232	145	110	107	152	209	46	43	216
20	239	234	21	84	175	170	85	36	223	218	37	116	143	138	117
238	17	24	235	174	81	88	171	222	33	40	219	142	113	120	139
23	236	237	18	87	172	173	82	39	220	221	34	119	140	141	114
233	22	19	240	169	86	83	176	217	38	35	224	137	118	115	144
68	191	186	69	4	255	250	5	100	159	154	101	12	247	242	13
190	65	72	187	254	1	8	251	158	97	104	155	246	9	16	243
71	188	189	66	7	252	253	2	103	156	157	98	15	244	245	10
185	70	67	192	249	6	3	256	153	102	99	160	241	14	11	248

	10			5			9					1			
	7			13			3					11			
	14			6			12					2			
	8			16			4					15			

Commentary in $\mathcal{D}$. Square $n = 16$, with 4×4 subsquares. It is noted that the 8×8 and 12×12 subsquares formed by associating contiguous 4×4 subsquares are magic as well.

قلا	قصح	قكح	قصو	قص	قلب	نط	رع	نو	رئح	رسب	س	قنط	قف	قمو	قعح	قعب	قن
قنط	قَم	قفز	قنب	قما	قلو	رسا	رئح	رنط	رند	سط	سد	قعا	قنح	قسط	قسد	قنط	قند
قلد	قفو	قلز	قمد	قنح	قصا	سب	رئح	سه	عب	رنه	رئح	قنب	قنح	قنه	قنب	قسه	قعح
قلج	قمح	ققد	قفه	قلح	قصب	سا	عا	رنو	رنز	سو	رسد	قنا	قسا	قسو	قنز	قنو	قعد
قصه	قفا	قنب	قلط	قنح	قل	رنز	رئح	ع	سز	رس	نح	قنح	قنح	قس	قنز	قع	قمح
قصح	قكز	قنصر	قكط	قله	قصد	سه	نه	رسط	نز	سج	رسو	قعه	قنه	قنط	قنز	قنح	قعو
كج	شو	ك	شد	رصح	كد	قيج	ريو	قي	ريد	رح	قيد	ما	رغ	لح	رفو	رف	مب
رصر	لب	رعه	رص	لج	كح	رز	قكب	ره	ر	قكج	قيح	رعط	ن	رعز	رعب	نا	مو
كو	رصد	كط	لو	رصا	رسط	قيو	رد	قيط	قكو	را	رط	مد	رعو	مز	ند	رعج	رفا
كه	له	رصب	رصح	ل	ش	قيه	فكه	رب	رج	فك	ري	مح	نح	رعد	رعه	مح	رغب
شج	رط	لد	لا	رصو	كب	رئح	قسط	فكد	فكا	رو	قيب	رعه	رعا	نب	مط	رعح	م
شا	يط	شه	كا	كز	شب	ريا	قط	ريه	قيا	قيز	ريب	رفح	لز	رفز	لط	مه	رفد
عز	رنب	عد	رن	رمد	عح	ه	شكد	ب	شكب	شيو	و	صه	رلد	صب	رلب	ركو	صو
رمج	فو	رما	رلو	فز	فب	شيه	يد	شيج	شع	يه	ي	ركه	قد	رئح	رئح	قه	ق
ف	رم	رغ	ص	رلز	رهمه	ح	شيب	يا	رئح	شط	شيز	صح	ركب	قا	رغ	رئح	ركز
عط	فط	رئح	رلط	فد	رمو	ز	يز	شي	شيا	يب	شيع	صز	قز	رك	ركا	قنب	رئح
رمط	رله	رغ	فه	رمب	عو	شكا	شز	يو	رئح	شيد	د	رلا	رئح	قو	رئح	ركد	صد
رمز	عج	رنا	عه	فا	رمح	شيط	ا	شكج	ج	ط	شك	رط	صا	رئح	صح	صط	رل

(18 × 18) $\mathcal{D}$: 98^r.

لوح ثمانية عشر متساوي الجمل من جميع جهاته وجملة كل صف من صفوفه ٢٩٢٥. وقد انقسم *tit.*: بتسعة مسطحات محلقة كل مسط منها من ضرب ستة في ستة وجملاها من جميع جوانبها متساوية وكل صف من صفوفها ٩٧٥. وكذلك كل مسط من مسطتين متصلين وهو اثنا عشر في اثني عشر متوافق ايضاً كل صف من صفوفه ١٩٥٠.

٢١٣. [(10, 5) شيج || ٢٠٥ (16, 7) شه || قعح [(14, 14) قفح || قلز [(13, 17) قلو || فلب [(13, 18) قلب

131	198	128	196	190	132	59	270	56	268	262	60	149	180	146	178	172	150
189	140	187	182	141	136	261	68	259	254	69	64	171	158	169	164	159	154
134	186	137	144	183	191	62	258	65	72	255	263	152	168	155	162	165	173
133	143	184	185	138	192	61	71	256	257	66	264	151	161	166	167	156	174
195	181	142	139	188	130	267	253	70	67	260	58	177	163	160	157	170	148
193	127	197	129	135	194	265	55	269	57	63	266	175	145	179	147	153	176
23	306	20	304	298	24	113	216	110	214	208	114	41	288	38	286	280	42
297	32	295	290	33	28	207	122	205	200	123	118	279	50	277	272	51	46
26	294	29	36	291	299	116	204	119	126	201	209	44	276	47	54	273	281
25	35	292	293	30	300	115	125	202	203	120	210	43	53	274	275	48	282
303	289	34	31	296	22	213	199	124	121	206	112	285	271	52	49	278	40
301	19	305	21	27	302	211	109	215	111	117	212	283	37	287	39	45	284
77	252	74	250	244	78	5	324	2	322	316	6	95	234	92	232	226	96
243	86	241	236	87	82	315	14	313	308	15	10	225	104	223	218	105	100
80	240	83	90	237	245	8	312	11	18	309	317	98	222	101	108	219	227
79	89	238	239	84	246	7	17	310	311	12	318	97	107	220	221	102	228
249	235	88	85	242	76	321	307	16	13	314	4	231	217	106	103	224	94
247	73	251	75	81	248	319	1	323	3	9	320	229	91	233	93	99	230

		2			6					1							
		8			3					7							
		5			9					4							

v	$\cdot$	ii	$\cdot$	$\cdot$	vi
$\cdot$	iv	$\cdot$	$\cdot$	v	x
$viii$	$\cdot$	i	$viii$	$\cdot$	$\cdot$
vii	vii	$\cdot$	$\cdot$	ii	$\cdot$
$\cdot$	$\cdot$	vi	iii	$\cdot$	iv
$\cdot$	i	$\cdot$	iii	ix	$\cdot$

Commentary in $\mathcal{D}$. Square $n = 18$, with 6×6 subsquares. It is noted that the 12×12 squares comprising four adjacent squares are magic as well; the next figure has a similar remark. Observe that in both the subsquares are filled alternately (see above, p. 129).

قصو	رز	رب	قصر	صب	شيا	شو	صح	قصح	ريه	رى	قفط	فد	شيط	شيد	فه	قف	رڭ	ريج	قفا
رو	قصج	ر	رج	ثى	فط	صو	شز	ريد	قفه	قصب	ريا	شيح	فا	لخ	شيه	ركب	قعر	قفد	ريط
قصط	رد	ره	قصد	صه	ثح	شط	ص	قصا	ريب	ريج	قفو	فز	شيو	شيز	فب	قفج	رك	ركا	قعح
را	قصح	قصه	رح	شه	صد	صا	شيب	رط	قص	قفز	ريو	شيح	فو	لخ	شك	ريز	قنب	قط	ركد
عو	شكر	شكب	عز	قعب	رلا	ركو	قعج	سح	شله	شل	سط	قصد	رلط	رلد	قمه	س	شمج	شلاح	سا
شكو	عج	ف	شكج	رل	قسط	قعو	ركز	شلد	سه	عب	شلا	رلخ	قسا	قصح	رله	شمب	نز	سد	شلط
عط	شكد	شكه	عد	قعه	رڭ	رلط	قع	عا	شلب	شلاج	سو	قصر	رلو	راز	قصب	سج	شم	شما	نخ
شكا	عح	عه	شكح	ركه	قعد	قعا	رلب	شكط	ع	سز	شلو	رلج	قسو	قصح	رم	شلز	سب	نط	شمد
قنو	رمز	رمب	قز	نب	شنا	شمو	نخ	قمح	رنه	رن	قبط	مد	ششط	شند	مه	قم	سج	رنج	قما
رمو	قنج	قس	رمج	شن	مط	نو	شمز	رند	قمه	قنب	رنا	شنح	ما	مح	شنه	رسب	قاز	قمد	رلط
قنط	رمد	رمة	قند	نه	شمج	شمط	ن	قنا	رنب	رنج	قمو	مز	شنو	شنز	مب	قمج	رس	رسا	قلح
رما	قنح	قنه	رمح	شمه	ند	نا	شذب	رمط	قن	قمر	رنو	شنج	مو	مح	شس	رنز	قنب	قلط	رسد
لو	شسر	شسب	لز	قلب	رعا	رسو	قلج	كح	شعه	شع	كط	قكد	رعط	رعد	فكه	ك	شفج	شعح	كا
شسو	لج	م	شسج	رع	فكط	قلو	رسز	شعد	كه	لب	شعا	رعح	فكا	فكح	رعه	شغب	يز	كد	شعط
لط	شسد	شسه	لد	قله	رمج	رسط	قل	لا	شعب	شعج	كو	قكر	رعو	رعز	قكب	كج	شف	شفا	يج
شسا	لح	له	شسح	رسه	قلد	قلا	رعب	شسط	ل	كز	شعو	رعج	قكو	فكج	رف	شعز	كب	يط	ششد
قبو	رفز	رفب	قيز	يب	شصا	شفو	يج	قح	رصه	رص	قط	د	شصط	شصد	ه	ق	شج	رصح	قا
رفو	قيج	قك	رفج	شص	ط	يو	شغز	رصد	قه	قيب	رصا	شصح	ا	ح	شصه	شب	صز	قد	رصط
قيط	رفد	رفه	قيد	يه	شفح	شغط	ى	قيا	رصب	رصح	قو	ز	شصو	شصز	ب	لج	ش	شا	صح
رفا	قيح	قيه	لغ	شفه	يد	يا	شصب	رط	قى	قز	رصو	شصح	و	ج	ت	رصز	قب	صط	شد

(20 × 20) D : 98^v.

لوح عشرين في عشرين كل صف منه ٤٠١٠. وينقسم بخمسة وعشرين مسطًا كل مسط منها : tit. منفصلًا باربعة في اربعة متفق الجمل ايضًا كل صف من كل مسط منها ٨٠٢. ومتى جمع اربعة مسطات منها حتى تكون مربع ثمانية في ثمانية اتفقت جملة ايضًا وكان كل صف ١٦٠٤. وكذلك ان جمع منها تسعة او ستة عشر يكون المجتمع اثني عشر في اثني عشر او ستة عشر في ستة عشر اتفقت الجمل ايضًا على حسب ما اشرنا اليه.

|| ٧ || (12, 14) عا || ٢٦ || (9, 14) سو || ٢١٢ || (13, 17) شيب || قصا || (20, 18) قسط || قعح || (12, 20) قفح ||
 قفب || (3, 1) قب || ١٤ || (13, 4) يج || ٥٢ || (13, 9) شنب || قنو quasi || (17, 12) قز

196	207	202	197	92	311	306	93	188	215	210	189	84	319	314	85	180	223	218	181
206	193	200	203	310	89	96	307	214	185	192	211	318	81	88	315	222	177	184	219
199	204	205	194	95	308	309	90	191	212	213	186	87	316	317	82	183	220	221	178
201	198	195	208	305	94	91	312	209	190	187	216	313	86	83	320	217	182	179	224
76	327	322	77	172	231	226	173	68	335	330	69	164	239	234	165	60	343	338	61
326	73	80	323	230	169	176	227	334	65	72	331	238	161	168	235	342	57	64	339
79	324	325	74	175	228	229	170	71	332	333	66	167	236	237	162	63	340	341	58
321	78	75	328	225	174	171	232	329	70	67	336	233	166	163	240	337	62	59	344
156	247	242	157	52	351	346	53	148	255	250	149	44	359	354	45	140	263	258	141
246	153	160	243	350	49	56	347	254	145	152	251	358	41	48	355	262	137	144	259
159	244	245	154	55	348	349	50	151	252	253	146	47	356	357	42	143	260	261	138
241	158	155	248	345	54	51	352	249	150	147	256	353	46	43	360	257	142	139	264
36	367	362	37	132	271	266	133	28	375	370	29	124	279	274	125	20	383	378	21
366	33	40	363	270	129	136	267	374	25	32	371	278	121	128	275	382	17	24	379
39	364	365	34	135	268	269	130	31	372	373	26	127	276	277	122	23	380	381	18
361	38	35	368	265	134	131	272	369	30	27	376	273	126	123	280	377	22	19	384
116	287	282	117	12	391	386	13	108	295	290	109	4	399	394	5	100	303	298	101
286	113	120	283	390	9	16	387	294	105	112	291	398	1	8	395	302	97	104	299
119	284	285	114	15	388	389	10	111	292	293	106	7	396	397	2	103	300	301	98
281	118	115	288	385	14	11	392	289	110	107	296	393	6	3	400	297	102	99	304

	1		14		2		15		3										
	16		4		17		5		18										
	6		19		7		20		8										
	21		9		22		10		23										
	11		24		12		25		13										

Commentary in $\mathcal{D}$. Square $n = 20$, divided into twenty-five 4×4 subsquares, with 8×8 , 12×12 and 16×16 subsquares also magic.

عح	فكا	قيو	عط	ح	فو	قيج	فخ	فز	قفط	صد	قه	ق	صه
فك	عه	فب	فيز	قص	قيب	فخ	ص	قط	ز	قد	صا	صح	قا
فا	قيح	قيط	عو	قصا	فط	قي	قيا	فد	و	صز	قب	فخ	صب
قيه	ف	عز	فكب	ه	قز	فخ	فه	قيد	فصب	صط	صو	صح	قو
يب	قفو	قفز	ط	كا	قف	يخ	قعح	قعب	كب	يو	قنب	قفح	يخ
سب	قلز	قلب	سبح	قعا	ل	قسط	قسد	لا	كو	ع	فكط	فكد	عا
قلو	نط	سو	قلج	كد	قصح	كز	لد	قسه	قعج	فكح	سز	عد	فكه
سه	قلد	قله	س	كج	لج	قسو	قسز	كح	قعد	عج	فكو	فكز	سبح
قلا	سد	سا	قلح	قعر	قصح	لب	كط	قع	ك	فكج	عب	سط	قل
قفه	يا	ي	قفح	قعه	يز	قعط	يط	كه	قعو	قفا	يه	يد	ققد
لح	قسا	قنو	لط	د	مو	قنج	قمح	مز	قصج	ند	قمه	قم	نه
قس	له	مب	قنز	قصد	قنب	مج	ن	قمط	ج	قمد	نا	نح	قما
ما	قنح	قنط	لو	قصه	مط	قن	قنا	مد	ب	نز	قنب	قج	نب
قنه	م	لز	قنب	ا	قز	مح	مه	قند	قصو	قلط	نو	نح	قمو

(14 × 14) $\mathcal{D}$: 102^f.

لوح اربعة عشر كل صف منه ١٣٧٩. وفي وسطه مربع ستة محقق كل صف منه ٥٩١. وفي اربع : *tit.*
 زوايا اربعة مربعات متوافقة الجمل فكذلك بين كل اثنين مثل كل واحد منهما كل صف من كل واحد
 ٣٩٤.

quod iterum ب *pr. scr., mut. in* ب [(12,5) ي || ب *et postea* سب *corr. ut vid. ex* [(14,10) يب
corr.

Division into unequal parts

78	121	116	79	8	86	113	108	87	189	94	105	100	95
120	75	82	117	190	112	83	90	109	7	104	91	98	101
81	118	119	76	191	89	110	111	84	6	97	102	103	92
115	80	77	122	5	107	88	85	114	192	99	96	93	106
12	186	187	9	21	180	18	178	172	22	16	182	183	13
62	137	132	63	171	30	169	164	31	26	70	129	124	71
136	59	66	133	24	168	27	34	165	173	128	67	74	125
65	134	135	60	23	33	166	167	28	174	73	126	127	68
131	64	61	138	177	163	32	29	170	20	123	72	69	130
185	11	10	188	175	17	179	19	25	176	181	15	14	184
38	161	156	39	4	46	153	148	47	193	54	145	140	55
160	35	42	157	194	152	43	50	149	3	144	51	58	141
41	158	159	36	195	49	150	151	44	2	57	142	143	52
155	40	37	162	1	147	48	45	154	196	139	56	53	146

	3		1		2		1		1				
			2				2						
	11										10		
	5				9						4		
	11										10		
	8		1		7		1		6				
			3				3						

<i>v</i>	$\cdot$	<i>ii</i>	$\cdot$	$\cdot$	<i>vi</i>
$\cdot$	<i>iv</i>	$\cdot$	$\cdot$	<i>v</i>	<i>x</i>
<i>viii</i>	$\cdot$	<i>i</i>	<i>viii</i>	$\cdot$	$\cdot$
<i>vii</i>	<i>vii</i>	$\cdot$	$\cdot$	<i>ii</i>	$\cdot$
$\cdot$	$\cdot$	<i>vi</i>	<i>iii</i>	$\cdot$	<i>iv</i>
$\cdot$	<i>i</i>	$\cdot$	<i>iii</i>	<i>ix</i>	$\cdot$

Commentary in $\mathcal{D}$. Square $n = 14$, with one central 6×6 square, four 4×4 subsquares in the corners, separated by 4×4 subsquares (and strips).

صز	قسد	صد	قشب	قنو	صح	لو	ركج	ريج	لز	قيه	قمو	قيب	قمد	قلح	قيو
قنه	قو	قنج	قمح	قز	قب	ركب	لح	م	ريط	قلز	قكد	قله	قل	قكه	قك
ق	قنب	قنج	قي	قبط	قنز	لظ	رك	ركا	لد	قيح	قلد	قكا	قكح	قلا	قلط
صط	قط	قن	قنا	قد	قنح	ريز	لح	له	ركد	قيز	قكز	قلب	قلج	قكب	قم
قسا	قمز	قنج	قه	قند	صو	يو	رمب	رمج	بيج	قمج	قكط	قكو	قكج	قلو	قيد
قنط	صج	قسج	صه	قا	قس	رما	يه	يد	رمد	قما	قيا	قمه	قيج	قيط	قنب
كح	رلا	ركو	كط	ح	رمط	لغ	قعا	قسو	فط	يب	رمه	مد	ريه	رى	مه
رل	كه	لب	ركز	رن	ز	قع	فه	صب	قنز	رمو	يا	ريد	ما	مح	ريا
لا	لج	ركط	كو	رنا	و	صا	قسح	قسط	فو	رمز	ى	مز	ريب	ريج	مب
ركه	ل	كز	رلب	ه	رنب	قسه	ص	فز	قعب	ط	رمح	رط	مو	مح	ريو
نحج	رح	ن	رو	ر	ند	د	رند	رنه	ا	عا	قص	سح	قنح	قنب	عب
قسط	سب	قصر	قصب	سج	نح	نحج	ج	ب	رنو	قفا	ف	قعط	قعد	فا	عو
نو	قصو	نط	سو	قصح	را	ك	رلط	رلد	كا	عد	قعح	عز	فد	قه	قفج
نه	سه	قصد	قصه	س	رب	لح	يز	كد	رله	عج	لغ	قعو	قعر	عح	ققد
ره	قضا	سد	سا	قصح	نب	كج	رلو	رلز	يج	قفز	قعج	فب	عط	قف	ع
رج	مط	رز	نا	نز	رد	لج	كب	يط	رم	قفه	سز	قنط	سط	عه	قنو

(16 × 16) $\mathcal{D}$: 103^f.

لوح ستة عشر متساوى الجمل كل صف من صفوفه ٢٠٥٦. وفي اربع زواياه اربعة مربعات : *tit.* محلقة كل مربع من ضرب ستة في ستة وجمل صفوف تلك المربعات متساوية من جميع جهاتها وجملة كل صف منها ٧٢١. وفي وسطه واربعة اضلاعه خمسة مربعات كل مربع منها اربعة في اربعة متساوى الجمل ايضا جملة كل صف منها ٥١٤.

مشرح. ذو فواصل ولب ومربعات وصليب خيالى ينضم بعضه مع المربع الذى فى الزاوية ويحصل له وفق مساو للذى ينضم منه مع الذى فى الزاوية الاخرى ومجموع كل نظيرين من الصليب ٢٥٧.

ut قفف [(2, 4) قه || قعج [(1, 4) قفج || ٣٣٢ [(13, 7) رلب || قه [(12, 13) قد || ٣١٩ [(7, 15) ريط
ut vid. قمو [(1, 1) قفو || *ut vid.* قعد [(1, 3) ققد || *vid.*

97	164	94	162	156	98	36	223	218	37	115	146	112	144	138	116
155	106	153	148	107	102	222	33	40	219	137	124	135	130	125	120
100	152	103	110	149	157	39	220	221	34	118	134	121	128	131	139
99	109	150	151	104	158	217	38	35	224	117	127	132	133	122	140
161	147	108	105	154	96	16	242	243	13	143	129	126	123	136	114
159	93	163	95	101	160	241	15	14	244	141	111	145	113	119	142
28	231	226	29	8	249	88	171	166	89	12	245	44	215	210	45
230	25	32	227	250	7	170	85	92	167	246	11	214	41	48	211
31	228	229	26	251	6	91	168	169	86	247	10	47	212	213	42
225	30	27	232	5	252	165	90	87	172	9	248	209	46	43	216
53	208	50	206	200	54	4	254	255	1	71	190	68	188	182	72
199	62	197	192	63	58	253	3	2	256	181	80	179	174	81	76
56	196	59	66	193	201	20	239	234	21	74	178	77	84	175	183
55	65	194	195	60	202	238	17	24	235	73	83	176	177	78	184
205	191	64	61	198	52	23	236	237	18	187	173	82	79	180	70
203	49	207	51	57	204	233	22	19	240	185	67	189	69	75	186

					7										
		2												1	
						10									
		8		12		3		11						6	
						13									
		5													
						9									

<i>v</i>	<i>·</i>	<i>ii</i>	<i>·</i>	<i>·</i>	<i>vi</i>
<i>·</i>	<i>iv</i>	<i>·</i>	<i>·</i>	<i>v</i>	<i>x</i>
<i>viii</i>	<i>·</i>	<i>i</i>	<i>viii</i>	<i>·</i>	<i>·</i>
<i>vii</i>	<i>vii</i>	<i>·</i>	<i>·</i>	<i>ii</i>	<i>·</i>
<i>·</i>	<i>·</i>	<i>vi</i>	<i>iii</i>	<i>·</i>	<i>iv</i>
<i>·</i>	<i>i</i>	<i>·</i>	<i>iii</i>	<i>ix</i>	<i>·</i>

Commentary in D. Square $n = 16$, with four 6×6 subsquares in the corners, separated by 4×4 subsquares (one central, *labb*) and 2×4 rectangles (*fawāṣil*). But, unlike what is said, extending the 6×6 corner squares to 10×10 squares will not satisfy the (usual) magic conditions (one diagonal will not be magic). Similar assertions are found in the third part, that of the central cross (pp. 154–161).

كح	رلا	ركو	كط	فد	قعه	قع	فه	لو	ركج	ريج	لز	صب	قصر	قشب	صج
رل	كه	لب	ركز	قعد	فا	فخ	قعا	ركب	لج	م	ربط	قسو	فط	صو	قسج
لا	ركج	ركط	كو	فز	قعب	قعج	فب	لط	رك	ركا	لد	صه	قسد	قسه	ص
رکه	ل	کز	رلب	قسط	فو	فخ	قعو	ريز	لح	له	رکد	قسا	صد	صا	قسح
عو	قفج	قعح	عز	قج	قی	قط	قشط	قن	قنا	قنب	قد	مد	ريه	ری	مه
قشب	عج	ف	قعط	قب	قيه	قمو	قيب	قمد	قلح	قيو	قنه	ريد	ما	مح	ريا
عط	قف	قفا	عد	قنو	قلز	قكد	قله	قل	فكه	قك	قا	مز	ريب	ريج	مب
قعر	عح	عه	قفد	ق	قبح	قلد	فكا	فكح	قلا	قلط	قنز	رط	مو	مح	ريو
ك	رلط	رلد	كا	قنح	قيز	قكر	قلب	قلج	فكب	قم	صط	نب	رز	رب	نح
رلح	يز	كد	رله	قنط	قمج	قكط	فكو	فكج	قلو	قيد	صح	رو	مط	نو	رج
كح	رلو	رلز	يح	صز	قما	قيا	قمه	قيج	قيط	قنب	قس	نه	رد	ره	ن
رلج	كب	يط	رم	قنح	قمز	قمح	فخ	قز	قو	قه	قند	را	ند	نا	رح
سح	قصا	قفو	سط	د	رنه	رن	ه	س	قصط	قصد	سا	يب	رمز	رمب	يخ
قص	سه	عب	قفز	رند	ا	ح	رنا	قصح	نز	سد	قصه	رمو	ط	يو	رمج
عا	قفح	قفط	سو	ز	رنب	ريج	ب	سج	قصو	قصز	نح	يه	رمد	رمة	ی
قفه	ع	سز	قصب	رملط	و	ج	رنو	قصج	سب	نط	ر	رما	يد	يا	رمح

(16 × 16) $\mathcal{D}$: 103^v.

لوح ستّة عشر متساوی الجمل من جميع جهاته كلّ صفّ من صفوفه ٢٠٥٦. وفي وسطه مربّع *tit.*:
 من ضرب ثمانية في ثمانية محلّق متساوی الجمل كلّ صفّ منه ١٠٢٨. ويحيط به اثنا عشر مربّعاً
 كلّ مربّع منها اربعة في اربعة متساوية الجمل كلّ صفّ منها ٥١٤.
 مشرح. ذو لبّ ومربّعات متساوية.

رلد [13, 7] رله || (v. seq.) ١٢٩ [9, 7] فكو || قفه [11, 16] قعه || فع [10, 16] قع

28	231	226	29	84	175	170	85	36	223	218	37	92	167	162	93
230	25	32	227	174	81	88	171	222	33	40	219	166	89	96	163
31	228	229	26	87	172	173	82	39	220	221	34	95	164	165	90
225	30	27	232	169	86	83	176	217	38	35	224	161	94	91	168
76	183	178	77	103	110	109	149	150	151	152	104	44	215	210	45
182	73	80	179	102	115	146	112	144	138	116	155	214	41	48	211
79	180	181	74	156	137	124	135	130	125	120	101	47	212	213	42
177	78	75	184	100	118	134	121	128	131	139	157	209	46	43	216
20	239	234	21	158	117	127	132	133	122	140	99	52	207	202	53
238	17	24	235	159	143	129	126	123	136	114	98	206	49	56	203
23	236	237	18	97	141	111	145	113	119	142	160	55	204	205	50
233	22	19	240	153	147	148	108	107	106	105	154	201	54	51	208
68	191	186	69	4	255	250	5	60	199	194	61	12	247	242	13
190	65	72	187	254	1	8	251	198	57	64	195	246	9	16	243
71	188	189	66	7	252	253	2	63	196	197	58	15	244	245	10
185	70	67	192	249	6	3	256	193	62	59	200	241	14	11	248

	10		3		9		2								
	4													8	
					1										
	11													7	
	5		13		6		12								

vii	xiv	xiii	.	.	.	.	viii
vi	v	.	ii	.	.	vi	.
.	.	iv	.	.	v	x	v
iv	viii	.	i	viii	.	.	.
.	vii	vii	.	.	ii	.	iii
.	.	.	vi	iii	.	iv	ii
i	.	i	.	iii	ix	.	.
.	.	.	xii	xi	x	ix	.

Commentary in $\mathcal{D}$. Square $n = 16$, with one central 8×8 square surrounded by twelve 4×4 subsquares.

ص	ب	ر	ل	ص	ي	م	د	ر	ف	ع	م	ق	ر	ك	ق	ش	ق	ر	ي	ق
ر	ل	ف	ص	ر	ل	م	ف	ر	ع	م	ر	ك	ق	ص	ق	ي	ر	ق	ف	ي
ص	ر	ل	ص	ي	م	د	ر	ف	ع	م	ق	ر	ك	ق	ش	ق	ر	ي	ق	ف
ر	ك	ق	ص	ر	ل	م	ف	ر	ع	م	ر	ك	ق	ص	ق	ي	ر	ق	ف	ي
ي	ب	ش	ي	ط	ك	ي	ق	ر	ي	ق	ز	ر	ف	ق	ص	ك	ح	ش	ي	ط
ف	د	ر	م	ل	ف	ق	ل	ق	م	ق	ف	ق	ف	ق	ق	ل	ل	ر	ف	ل
ر	م	ف	ا	ل	ك	ق	ل	ق	م	ق	ف	ق	ق	ق	ق	ل	ل	م	ف	ل
ف	ز	م	ر	ا	ب	ف	ق	ص	ق	م	ق	ق	ق	ق	ق	ل	ل	ر	ف	ل
ر	ل	ز	ف	ل	م	د	ر	ف	ع	م	ق	ر	ك	ق	ش	ق	ر	ي	ق	ف
ك	ر	ص	ر	ك	ق	ص	ق	ل	ق	م	ق	ق	ق	ق	ق	ل	ل	م	ف	ل
ر	ص	ك	ل	ب	ر	ه	ق	ص	ق	م	ق	ق	ق	ق	ق	ل	ل	م	ف	ل
لا	ر	ص	ر	ك	ق	ص	ق	ل	ق	م	ق	ق	ق	ق	ق	ل	ل	م	ف	ل
ر	ص	ل	ك	ز	ش	ر	ق	ف	ق	م	ق	ق	ق	ق	ق	ل	ل	م	ف	ل
ش	ي	ا	ي	ش	ي	ق	ر	ا	ي	ق	ر	ق	ق	ق	ق	ل	ل	م	ف	ل
ع	و	ر	ا	م	ع	ر	ن	د	ر	ن	س	ك	ش	ك	س	ر	س	ر	س	س
ر	ن	ع	ف	ر	م	ش	ك	ن	ع	ب	ر	ش	و	ي	ج	س	و	ن	س	س
ع	ط	ر	م	ع	ك	ع	ا	ر	و	ز	س	ك	ش	ه	ي	ب	س	ر	س	س
ر	م	ع	ع	ه	ر	ب	ا	ن	ج	ع	ز	س	ا	ك	ب	ي	ط	ش	ك	س

(18 × 18) $\mathcal{D} : 104^v$.

لوح ثمانية عشر متساوي الجمل جملة كل صف منه ٢٩٢٥. وفي وسطه مربع مَحَلَّق من *tit.*: ضرب عشرة في عشرة متساوي الجمل من جميع جهاته كل صف منه ١٦٢٥. وفي اربع زواياه واربعة اضلاعه اثنا عشر مربعًا كل واحد من ضرب اربعة في اربعة متساوي الجمل من جميع جهاته وجملة كل صف من صفوفه ٦٥٠.

مشرح. ذو لب وفواصل ومربعات متساوية مجموع كل نظيرين من الفاصلة ٣٢٥.

٢٠٢. [(7, 4) ش ب || ١٣١ [(14, 14) ق ك || ٤٣ [(10, 16) م ب

92	235	230	93	16	44	283	278	45	100	227	222	101	309	108	219	214	109
234	89	96	231	310	282	41	48	279	226	97	104	223	15	218	105	112	215
95	232	233	90	311	47	280	281	42	103	224	225	98	14	111	216	217	106
229	94	91	236	13	277	46	43	284	221	102	99	228	312	213	110	107	220
12	314	315	9	121	212	114	210	117	206	200	127	196	122	8	318	319	5
84	243	238	85	195	137	144	143	183	184	185	186	138	130	36	291	286	37
242	81	88	239	128	136	149	180	146	178	172	150	189	197	290	33	40	287
87	240	241	82	199	190	171	158	169	164	159	154	135	126	39	288	289	34
237	86	83	244	124	134	152	168	155	162	165	173	191	201	285	38	35	292
28	299	294	29	123	192	151	161	166	167	156	174	133	202	52	275	270	53
298	25	32	295	205	193	177	163	160	157	170	148	132	120	274	49	56	271
31	296	297	26	118	131	175	145	179	147	153	176	194	207	55	272	273	50
293	30	27	300	209	187	181	182	142	141	140	139	188	116	269	54	51	276
313	11	10	316	203	113	211	115	208	119	125	198	129	204	317	7	6	320
76	251	246	77	4	68	259	254	69	20	307	302	21	321	60	267	262	61
250	73	80	247	322	258	65	72	255	306	17	24	303	3	266	57	64	263
79	248	249	74	323	71	256	257	66	23	304	305	18	2	63	264	265	58
245	78	75	252	1	253	70	67	260	301	22	19	308	324	261	62	59	268

	4		1					3		1				2			
		4		10													
	15														15		
	5															11	
							1										
	12															9	
	15																15
	6		1			7			13		1				8		
		6															

<i>ix</i>	.	<i>ii</i>	.	<i>v</i>	.	.	<i>xv</i>	.	<i>x</i>
.	<i>vii</i>	<i>xiv</i>	<i>xiii</i>	.	.	.	.	<i>viii</i>	<i>xviii</i>
<i>xvi</i>	<i>vi</i>	<i>v</i>	.	<i>ii</i>	.	.	<i>vi</i>	.	.
.	.	.	<i>iv</i>	.	.	<i>v</i>	<i>x</i>	<i>v</i>	<i>xiv</i>
<i>xii</i>	<i>iv</i>	<i>viii</i>	.	<i>i</i>	<i>viii</i>	.	.	.	.
<i>xi</i>	.	<i>vii</i>	<i>vii</i>	.	.	<i>ii</i>	.	<i>iii</i>	.
.	.	.	.	<i>vi</i>	<i>iii</i>	.	<i>iv</i>	<i>ii</i>	<i>viii</i>
<i>vi</i>	<i>i</i>	.	<i>i</i>	.	<i>iii</i>	<i>ix</i>	.	.	.
.	.	.	.	<i>xii</i>	<i>xi</i>	<i>x</i>	<i>ix</i>	.	<i>iv</i>
.	<i>i</i>	.	<i>iii</i>	.	<i>vii</i>	<i>xiii</i>	.	<i>xvii</i>	.

Commentary in D. Square $n = 18$, with one central 10×10 square, and twelve corner and lateral 4×4 subsquares with separating strips.

قمح	قن	قسط	رنج	رند	رنه	رنو	قمد	كح	شعه	شع	كط	قعه	ققب	قفا	ركا	ركب	ركج	ركد	قمو
قنب	قنه	رن	قنب	رمب	قنو	رنط	شعد	كه	لب	شعا	قعد	قفز	رنج	ققد	ريو	ري	قفتح	ركز	
رس	رما	قسد	رلط	رلد	قسه	قس	قما	لا	شعب	شعج	كو	ركج	رط	قصو	رز	رب	قصرز	قصب	قعج
قم	قنح	رلح	قسا	قصح	رله	رمج	رسا	شسط	ل	كز	شعو	قعب	قص	رو	قصح	ر	رج	ريا	ركط
رسب	قنز	قنز	رلو	راز	قشب	رمد	قلط	س	شعج	شلع	سا	رل	قنط	قسط	رد	ره	قصد	ريب	قعا
رمنج	رمز	رلج	قسو	قصح	رم	قند	قلح	شعب	نز	سد	شلط	رلا	ريه	را	قصح	قسه	رح	قفو	قع
قلز	رمة	قنا	رمط	قنح	قنط	رمو	رمد	سج	شم	شما	نح	قسط	رنج	قفتح	رنز	قفه	قضا	ريد	رلب
رنز	رنا	رنب	قنح	قنز	قمو	قنه	رنح	شاز	سب	نط	شمد	رکه	رلط	رك	قف	قسط	قعج	قنز	ركو
د	شسط	شصد	ه	مد	شنط	شند	مه	قلب	رعا	رسو	قلج	نب	شنا	ثمو	نح	ك	شفتح	شعج	كا
شصح	ا	ح	شعه	شنع	ما	مح	شنه	رع	قكط	قلو	رسز	شن	مط	نو	شز	شغب	يز	كد	شعط
ز	شصو	شعز	ب	مز	شنو	شنز	مب	قله	رصح	رسط	قل	نه	شمح	شسط	ن	كج	شف	شفا	نح
شصح	و	ج	ت	شنع	مو	مج	شس	رسه	قلد	قلا	رعب	شمه	ند	نا	شنب	شعز	كب	يط	شغد
عا	عح	عز	شكه	شكو	شكز	شكح	عب	لو	شسز	شسب	لز	قج	قي	قط	رصح	رصد	رصه	رصو	قد
ع	نح	شكب	ف	شك	شيد	فد	شلا	شسو	لج	م	شسج	قب	قيه	رص	قيب	نح	رفب	قيو	رصط
شلب	شسج	صب	شيا	شو	صصح	نح	سط	لط	شسد	شسه	لد	ش	رفا	قكد	رعط	رعد	فكه	فك	قا
سج	فو	شى	قط	صو	شز	شيه	شلع	شسا	لح	له	شسح	ق	قبح	رعح	قكا	قكح	رعه	رفج	شا
شلد	فه	صه	شع	شط	ص	شيو	سز	يب	شصا	شفو	نح	شب	قيز	قكر	رعو	رعز	قكب	رقد	صط
شله	شيط	شه	صد	صا	شيب	فب	سو	شص	ط	يو	شفز	شعج	رفز	رعج	قكو	قكج	رف	قيد	صح
سه	شيز	عط	شكا	فا	فز	شيع	شلو	يه	شفح	شفط	ى	صز	رفه	قيا	رط	قيج	قيط	رفو	شد
شكط	شكج	شكد	عو	عه	عد	عج	شل	شفه	يد	يا	شصب	رصز	رصا	رصب	نح	قز	قو	قه	رصح

(20 × 20) D : 105^v.

لوح عشرين في عشرين كل صف منه ٤٠١٠. وفي زواياها اربعة مربعات محلفة كل مربع منها tit. ثمانية في ثمانية متفقة الجمل كل صف منها ١٦٠٤. وفي اضلاعه اربعة مربعات كل مربع اربعة في اربعة متفقة الجمل كل صف منها ٨٠٢. وفي وسطه مربع اربعة في اربعة كل صف منه ٨٠٢. وفي خلاله اربعة مربعات كل واحد منها اربعة في اربعة متفقة الجمل كل صف منها ٨٠٢.

مشرح. ذو صليب ينقسم بمربعات متساوية ذوات وفق ومربعات في زواياها متساوية متوافقة ايضاً.

[(2,9) يط || ut vid. شفط (1,11) شعط || شمه (9,13) شمد || ٣٦٤ (13,14) رسد || ٤٤ (13,20) قمد
٣٦ (13,2) شلو || ٢٦٣ (5,8) رصح || قله (11,9) قلد || ut vid. شط

قند	رنط	رند	تمه	کخ	شعه	شع	کط	قنب	رنا	رمو	قنچ	صو	شز	شب	صز	قس	رمج	رلخ	قسا
رنج	قما	قمح	رنه	شعد	که	لب	شعا	رن	قط	قنو	رمز	شو	صج	ق	شج	رمب	قنز	قسد	رلط
قمز	رنو	رنز	قنب	لا	شعب	شعج	کو	قنه	رمط	قن	صط	شد	شه	صد	قسج	رم	رما	قنح	
رنج	قمو	قج	رس	شسط	ل	کز	شعو	رमे	قند	قنا	رنب	شا	صع	صه	شخ	رلز	قنب	قنط	رمد
ک	شفج	شعج	کا	عا	شلد	سج	شلب	شکو	عب	قنز	رنج	ققد	ریو	ری	قفح	لخ	شیه	شی	فط
شغب	یز	کد	شعط	شکه	ف	شکج	شیح	فا	عو	رط	قصو	رز	رب	قصر	قصب	شید	فه	صب	شیا
کج	شف	شفا	یج	عد	شکب	عز	فد	شیط	شکز	قص	رو	قصج	ر	رج	ریا	صا	شیب	شیج	فو
شعر	کب	یط	ششد	عج	لخ	شک	شکا	عح	شکح	قفط	قصط	رد	ره	قصد	ریب	شط	ص	فز	شیو
قیب	رصا	رفو	قیج	شلا	شیز	فب	عط	شکد	ع	ریه	را	قصح	قصه	رح	قفو	قلو	رنز	رسب	قلز
رص	قط	قیو	رفز	شکط	سز	شلج	سط	عه	شل	رنج	قفج	ریز	قفه	قصا	رید	رسو	قلج	قم	رنج
قیه	لخ	رط	قی	قسط	رلو	قسو	رلد	رک	قع	نچ	شنب	ن	شن	شمد	ند	قلط	رسد	رسه	قلد
رفه	قید	قیا	رصب	رکز	قعح	رکه	رک	قعد	شج	سب	شما	شلو	سج	نخ	رسا	قلح	قله	رنح	
مد	شنط	شند	مه	قعب	رکد	قعه	قنب	رکا	رکط	نو	شم	نط	سو	شلز	شمه	یب	شصا	شفو	یج
شنح	ما	مح	شنه	قما	قفا	رکب	رک	قعو	رل	نه	سه	شلح	شلاط	س	شمو	شص	ط	یو	شفز
مز	شنو	شنز	مب	رلخ	ریط	قف	قعر	رکو	قسح	شط	شله	سد	سا	شطب	نب	یه	شفع	شفت	ی
شنج	مو	مح	شس	رلا	قسه	رله	قنر	قعج	رلب	شز	مط	شنا	نا	نز	شح	شفه	ید	یا	شصب
قد	رصط	رصد	قه	لو	شنز	شسب	لز	قک	لخ	رنح	قکا	د	شصط	شصد	ه	قکح	رعه	رع	قکط
رصح	قا	لخ	رصه	شسو	لج	م	شسج	رفب	قیز	قکد	رعط	شصح	ا	ح	شصه	رعد	قکه	قلب	رعا
قز	رصو	رصر	قب	لط	شمد	شسه	لد	قکج	رف	رفا	قیح	ز	شصو	شصز	ب	قلا	رعب	رعج	قکو
صج	قو	لخ	ش	شسا	لح	له	شسح	رعز	قکب	قیط	رقد	ثصح	و	ج	ت	رسط	قل	قکر	رعو

لوح عشرين في عشرين جملة كلّ صفّ من صفوفه ٤٠١٠. وفي وسطه مربع من ضرب اثني: *tit.* عشر في مثله متفق الجمل ايضاً كلّ صفّ من صفوفه ٢٤٠٦. وهذا المربع مقسوم باربعة مربعات محلّقة كلّ مربع سته في سته متّفقة الجمل كلّ صفّ من صفوفه ١٢٠٣. ويحيط بهذا المربع سته عشر مربّعاً كلّ واحد من ضرب اربعة في اربعة متّفقة الجمل ايضاً كلّ صفّ من صفوفه ٨٠٢. والله اعلم.

مشرح. ذو لبّ منقسم بمربّعات متساوية متوافقة.

(12, 19) رن || *ut vid.* (v. 9, 19) رمر || (7, 16) ريو || (15, 15) ف || يو || (14, 10) قسو || رسه || (3, 10) رسد || (v. *infra*) ٣١٧ || فغ || (v. *infra*) ٨٣ || (15, 14) شكب || (12, 1) رع || (12, 1) رعز || (1, 4) قكط || قيه || (19, 9) قيد || قصو

144	259	254	145	28	375	370	29	152	251	246	153	96	307	302	97	160	243	238	161
258	141	148	255	374	25	32	371	250	149	156	247	306	93	100	303	242	157	164	239
147	256	257	142	31	372	373	26	155	248	249	150	99	304	305	94	163	240	241	158
253	146	143	260	369	30	27	376	245	154	151	252	301	98	95	308	237	162	159	244
20	383	378	21	71	334	68	332	326	72	187	218	184	216	210	188	88	315	310	89
382	17	24	379	325	80	323	318	81	76	209	196	207	202	197	192	314	85	92	311
23	380	381	18	74	322	77	84	319	327	190	206	193	200	203	211	91	312	313	86
377	22	19	384	73	83	320	321	78	328	189	199	204	205	194	212	309	90	87	316
112	291	286	113	331	317	82	79	324	70	215	201	198	195	208	186	136	267	262	137
290	109	116	287	329	67	333	69	75	330	213	183	217	185	191	214	266	133	140	263
115	288	289	110	169	236	166	234	228	170	53	352	50	350	344	54	139	264	265	134
285	114	111	292	227	178	225	220	179	174	343	62	341	336	63	58	261	138	135	268
44	359	354	45	172	224	175	182	221	229	56	340	59	66	337	345	12	391	386	13
358	41	48	355	171	181	222	223	176	230	55	65	338	339	60	346	390	9	16	387
47	356	357	42	233	219	180	177	226	168	349	335	64	61	342	52	15	388	389	10
353	46	43	360	231	165	235	167	173	232	347	49	351	51	57	348	385	14	11	392
104	299	294	105	36	367	362	37	120	283	278	121	4	399	394	5	128	275	270	129
298	101	108	295	366	33	40	363	282	117	124	279	398	1	8	395	274	125	132	271
107	296	297	102	39	364	365	34	123	280	281	118	7	396	397	2	131	272	273	126
293	106	103	300	361	38	35	368	277	122	119	284	393	6	3	400	269	130	127	276

	5			17			4			11					3				
	18														12				
					13					1									
	9														6				
	15					2				14					19				
	19			16			8			20					7				

v	.	ii	.	.	vi
.	iv	.	.	v	x
viii	.	i	viii	.	.
vii	vii	.	.	ii	.
.	.	vi	iii	.	iv
.	i	.	iii	ix	.

Commentary in D. Square $n = 20$, with four 6×6 squares (not: one 12×12 , see p. 107*n*) in the centre surrounded by sixteen 4×4 subsquares.

قنا	رند	قمح	رنب	رمو	قنب	يب	قسط	رلو	قسو	رلد	رك	قع	شفط	قفز	ريخ	قند	ريو	رى	ققح
رمه	قس	رمج	رلخ	قسا	قنو	شص	ركز	قمح	ركه	رك	قعط	قعد	يا	رط	قصو	رز	رب	قصر	قصب
قند	رمب	قنز	قند	رلط	رمز	شصا	قعب	ركد	قعه	قنب	ركا	ركط	ى	قص	رو	قصح	ر	رج	ريا
قنج	قصح	رم	رما	قنج	رمح	ط	قعا	قفا	ركب	ركج	قعو	رل	شصب	قفط	قسط	رد	ره	قصد	ريب
رنا	راز	قصب	قنط	رمد	قن	ح	رلخ	ريط	قف	قنز	ركو	قصح	شصب	ريه	را	قصح	قصه	رح	قفو
رمط	قمز	رنج	قنط	قنه	رن	شصد	رلا	قسه	رله	قنز	قصح	رلب	ز	ريخ	قصح	ريز	قفه	قضا	ريد
كد	شعح	شعط	كا	ك	شغب	لو	شسر	شسب	لز	نب	شنا	شمو	نح	شفج	يز	يو	شفو	شفز	يخ
عط	شكو	عو	شكد	شبح	ف	شسو	لج	م	شسح	شن	مط	نو	شمز	قلج	رعب	قل	رع	رسد	قلد
شيز	فلخ	شيه	شى	فط	فد	لط	شسد	شسه	لد	نه	شمح	شط	ن	ريخ	قنب	رسا	رنو	قمح	قلح
فب	شيد	فه	صب	شيا	شيط	شسا	لح	له	شسح	شه	ند	نا	شذب	قلو	رس	قلط	قمو	رنز	رسه
فا	صا	شيب	شبح	فو	شك	مد	شسظ	شند	مه	كح	شعه	شع	كط	قله	قحه	رنح	رلط	قم	رسو
شكج	شط	ص	فز	شيو	عح	شنح	ما	مح	شنه	شعد	كه	لب	شعا	رسط	رنه	قمد	قما	رسب	قلب
شكا	عه	شكه	عز	فلخ	شكب	مز	شنو	شنز	مب	لا	شعب	شعج	كو	رسز	فكط	رعا	قلا	قلز	ريخ
شعز	كج	كب	شف	شفا	يط	شنج	مو	مح	شس	شسظ	ل	كز	شعو	يخ	شغد	شفه	يه	يد	شفح
سا	شمد	نح	شئب	شلو	سب	شصه	صز	شبح	صد	شو	ش	صع	و	قيه	رص	قيب	رفح	رفب	قيو
شله	ع	شلج	شكح	عا	سو	ه	رسط	قو	رصر	رصب	قز	قب	شصو	رفا	قكد	رعط	رعد	فكه	فك
سد	شلب	سز	عد	شكط	شلز	د	ق	رصو	قج	قى	رصح	شا	شصز	قيح	رعح	قكا	قكح	عه	رفج
سج	عج	شل	شلا	سح	شلح	شصح	صط	قط	رصد	رصد	قد	شب	ج	قيز	قكز	رعو	رعز	فكب	رفد
شما	شكز	عب	سط	شلد	س	شصط	شه	رصا	قح	قه	رصح	صو	ب	رفز	رعج	قكو	قكج	رف	قيد
شلاط	نز	شمج	نط	سه	ثم	ا	شبح	صح	شز	صه	قا	شد	ت	رفه	قيا	رفط	قيج	قيط	رفو

(20 × 20) $\mathcal{D}$: 106^v.

لوح عشرين في عشرين وحيلة كل صف من صفوفه ٤٠١٠. وفي وسطه مربع ثمانية في ثمانية : tit. متفق الجمل كل صف من صفوفه ١٦٠٤. وهذا المربع مقسوم باربعة مربعات كل واحد من ضرب اربعة في اربعة متفقة الجمل كل صف من صفوفها ٨٠٢. ويحيط بهذا المربع ثمانية مربعات محلقة اربعة في الزوايا واربعة في الاضلاع كل واحد منها من ضرب ستة في ستة متفقة الجمل ايضاً كل صف من صفوفها ١٢٠٣.

مشرح. ذو لب وفواصل ومربعات متساوية مجموع كل نظيرين من الفاصلة ٤٠١.

شعر quasi [(20, 7) شعر || شو [(19, 13) شكو || ٢٥٠ [(10, 13) شن || يد [(5, 14) يز

151	254	148	252	246	152	12	169	236	166	234	228	170	389	187	218	184	216	210	188
245	160	243	238	161	156	390	227	178	225	220	179	174	11	209	196	207	202	197	192
154	242	157	164	239	247	391	172	224	175	182	221	229	10	190	206	193	200	203	211
153	163	240	241	158	248	9	171	181	222	223	176	230	392	189	199	204	205	194	212
251	237	162	159	244	150	8	233	219	180	177	226	168	393	215	201	198	195	208	186
249	147	253	149	155	250	394	231	165	235	167	173	232	7	213	183	217	185	191	214
24	378	379	21	20	382	36	367	362	37	52	351	346	53	383	17	16	386	387	13
79	326	76	324	318	80	366	33	40	363	350	49	56	347	133	272	130	270	264	134
317	88	315	310	89	84	39	364	365	34	55	348	349	50	263	142	261	256	143	138
82	314	85	92	311	319	361	38	35	368	345	54	51	352	136	260	139	146	257	265
81	91	312	313	86	320	44	359	354	45	28	375	370	29	135	145	258	259	140	266
323	309	90	87	316	78	358	41	48	355	374	25	32	371	269	255	144	141	262	132
321	75	325	77	83	322	47	356	357	42	31	372	373	26	267	129	271	131	137	268
377	23	22	380	381	19	353	46	43	360	369	30	27	376	18	384	385	15	14	388
61	344	58	342	336	62	395	97	308	94	306	300	98	6	115	290	112	288	282	116
335	70	333	328	71	66	5	299	106	297	292	107	102	396	281	124	279	274	125	120
64	332	67	74	329	337	4	100	296	103	110	293	301	397	118	278	121	128	275	283
63	73	330	331	68	338	398	99	109	294	295	104	302	3	117	127	276	277	122	284
341	327	72	69	334	60	399	305	291	108	105	298	96	2	287	273	126	123	280	114
339	57	343	59	65	340	1	303	93	307	95	101	304	400	285	111	289	113	119	286

		3		1		2		1							1				
			4					4											
		13													13				
					11			9											
		7																4	
					10			12											
		13																13	
		8		1		6		1							5				
			4					4											

<i>v</i>	·	<i>ii</i>	·	·	<i>vi</i>
·	<i>iv</i>	·	·	<i>v</i>	<i>x</i>
<i>viii</i>	·	<i>i</i>	<i>viii</i>	·	·
<i>vii</i>	<i>vii</i>	·	·	<i>ii</i>	·
·	·	<i>vi</i>	<i>iii</i>	·	<i>iv</i>
·	<i>i</i>	·	<i>iii</i>	<i>ix</i>	·

Commentary in D. Square $n = 20$, with four 4×4 squares (or one 8×8) in the centre and eight surrounding 6×6 subsquares, separated by strips.

قسط	رلو	قسو	رلد	رك	قع	صب	شيا	شو	صج	شنج	مز	شنح	مد	قفز	رج	ققد	ريو	رى	قفع
ركز	قفع	ركه	رك	قسط	قعد	شى	فط	صو	شز	مو	شنو	ما	شنط	رط	قصو	رز	رب	قصر	قصب
قعب	ركد	قعه	قغب	ركا	ركط	صه	شح	شط	ص	مج	شز	مح	شند	قص	رو	قصج	ر	رج	ريا
قعا	قفا	ركب	رك	قعو	رل	شه	صد	صا	شيب	شس	مب	شنه	مه	قفط	قصط	رد	ره	قصد	ريب
رلج	رلج	قف	قعر	ركو	قصح	كه	شعه	شعد	كح	مط	شنا	شن	نب	ريه	را	قصح	قعه	رح	قفو
رلا	قسه	رله	قعر	قعج	رلب	شعو	كو	كز	شعج	شنب	ن	نا	شمتط	ريج	قفعج	ريز	قفه	قضا	ريد
لب	شعا	شسو	لج	شلز	سد	قلط	قمو	قعه	ررز	رنج	رنط	رس	قم	شسا	م	ف	شكج	شيع	فا
شع	كط	لو	شسر	سج	شلح	قلح	قنا	رند	قمج	رنب	رمو	قنب	سج	لط	شسب	شكب	عز	فد	شيط
له	ششح	شسط	ل	سب	شلاط	رسد	رمة	قس	رمج	رلح	قسا	قنو	قار	لح	ششح	لج	شك	شكا	عح
شسه	لد	لا	شعب	شم	سا	قلو	قند	رمب	قز	قسد	رلط	رمز	رسة	شسد	لز	شيز	فب	عط	شكد
شما	نط	شمو	نو	شفتط	يب	رسو	قنج	قصح	رم	رما	قنج	رمج	قله	شيع	لج	شفا	يط	شغو	يو
نح	شمد	نح	شمز	يا	شص	رسز	رنا	راز	قصب	قنط	رمد	قن	قلد	فز	شيد	يح	شغد	يح	شفز
نه	شمه	س	شعب	ى	شصا	قلج	رمط	قعر	ركج	قمط	قنه	رن	سج	فو	شيه	يه	شغه	ك	شغب
شمج	ند	شمج	نز	شصب	ط	رسا	رنه	رنو	قمد	قمج	قنب	قما	رسب	شيو	فه	شفج	يد	شفج	يز
قا	شد	صح	شب	رصو	قب	عج	شكز	شكو	عو	كا	شعط	شعج	كد	قيط	رفو	قيو	رقد	رعج	قك
رصه	قى	رصح	لج	قيا	قو	شكج	عد	عه	شكه	شف	كب	كج	شعز	رعز	ككج	عه	رع	قكط	قكد
قد	رصب	قز	قيد	رط	رصر	د	شصط	شصد	ه	شكط	عا	شلد	سج	قكب	رعد	قكه	قلب	رعا	رعط
لج	قيج	رص	رصا	لج	رصح	ا	ح	شصه	ع	شلب	سه	شله	قكا	قلا	رعب	رعج	قكو	رف	
شا	رفز	قيب	قط	رصد	ق	ز	شصو	شصز	ب	سز	شلع	عب	شل	رفج	رسط	قل	قكر	رعو	قيح
رسط	صز	شيع	صط	قه	ش	شصح	و	ج	ت	شلو	سو	شلا	سط	رفا	قيه	رفه	قيز	قكج	رفب

(20 × 20) D : 107^r.

لوح عشرين في عشرين وحلة كل صف من صفوفه ٤٠١٠. وفي وسطه مربعة ثمانية في tit.: ثمانية محلة متفقة الجمل وحلة كل صف من صفوفها ١٦٠٤. وفي اربع زواياها اربعة مربعات محلة كل واحد من ضرب ستة في ستة متفقة الجمل كل صف من صفوفها ١٢٠٣. وفي اربعة اضلاعه ثمانية مربعات كل واحد منها من ضرب اربعة في اربعة متفقة الجمل ايضا كل صف من صفوفها ٨٠٢.

مشكل مشرح. ذو لب وفواصل ومربعات محلة ومساوية بعضها لبعض ذوات جمل متساوية مجموع كل نظيرين من الفاصلة ٤٠١.

رقب [(1, 1) رقب || فج [(20, 3) رقب || رفو [(5, 6) رقب || ٣٨٣ [(1, 8) شغب || ٧٣ [(3, 11) فب

تمت الالواح التي اوردها الفضل بن ثابت من المختلطات في كتابه هاهنا : Expl. D, 107^v.

Excerpta e cod. D, 102^v :

فصل في الصلب أنّه يقع في شكل زوج الفرد في وسطه وهو سطران مقتّنان طولاً وسطران عرضاً فيقع في جوانبه الاربعة اربعة مربّعات كلّ مربّع زوج وكلّ صفّ من صفوفها متّفق الجمل. ويصير الصلب بينها مشتركاً فكلّ مربّع حاصل مع سطرى الصلب في الجهات الاربع متّفق الجمل ايضاً. واضلاع الاشكال التي تقع فيها الصلب $\bar{\text{ى}} \bar{\text{يد}} \bar{\text{يح}} \bar{\text{كب}} \bar{\text{كو}}$ متزايد اربعة اربعة ($sc. n = 4k + 2, k \geq 2$). ووجه معرفة مبدأ العدد الاول فيه من الصغار ان نأخذ نصف عدد الصلب فننقصه من عدد نصف المربّع ونزيد عليه واحداً ابداً فيخرج المبدأ $(1 + [\frac{n}{2} - \frac{1}{2}][4n - 4])$. مثاله في ضلع عشرة أخذنا نصف عدد الصلب وهو اربعة اسطر هاهنا $\bar{\text{آ}}$ اربعة ابداً والاربعة الاسطر تنتهى هاهنا الى $\bar{\text{م}}$ نقصنا منه اربعة ابداً يبقى لو نصفناه فيحصل $\bar{\text{يح}}$ نقصناه من نصف مربّع العشرة الذى هو $\bar{\text{ن}}$ فيبقى $\bar{\text{لب}}$ زدنا عليه واحداً ابداً فيصير $\bar{\text{لج}}$ وهو مبدأ الصلب في شكل العشرة. فنجرّيه كما في المثال هناك وثبت قرينه في سطره. وذلك ما اردنا ذكره.

كح	عه	ع	كط	م	سا	ك	لج	عح	كا
عد	كه	لب	عا	سب	لط	فب	يز	كد	عط
لا	عب	عج	كو	سج	لح	كج	ف	فا	بح
سط	ل	كز	عو	لز	سد	عز	كب	يط	فد
ما	نط	مج	نز	ن	نج	لو	سو	سز	لج
س	مب	نح	مد	مح	نا	سه	له	لد	سج
د	صط	صد	ه	مط	نب	يب	صا	فو	بج
صم	ا	ح	صه	ند	مز	ص	ط	يو	فز
ز	صو	صز	ب	مو	نه	يه	لخ	فط	ى
صم	و	ج	ق	نو	مه	فه	يد	يا	صب

(10 × 10) $\mathcal{D}$: 102^f.

لوح عشرة متساوى الصفوف كلّ صفّ منه ٥٠٥. وفي اربع زواياه اربعة مربّعات كلّ مربّع مسطّ *tit.*: كلّ صفّ من كلّ واحد ٢٠٢. واذا اضيف الى كلّ مسطّ بعض الصليب كان له وفق. ومن هاهنا <فيما بعد> من هذه المقالة <مما ذكره> المفصل بن ثابت ما وجدناه في كتابه.

ut vid. [مح (1,8)]

Cross in the middle

As indicated in MS. $\mathcal{D}$ after the above 20×20 square (see p. 152, bottom), we have reached the end of the examples in our treatise. Indeed, the four examples with the central cross, which we have chosen to present separately at the end (since they are treated last in the theory), are found among those displaying unequal parts, all these examples being in order of size. Note that this first example even heads the sequence of examples with unequal parts, whence the remark in MS. $\mathcal{D}$ that what follows ‘from here on’ is taken from al-Mufaḍḍal’s text.

As an introduction to the first case, the commentator indicates a way to find the smallest number in the $2 \times n$ cross when it is filled first: adding 1 to half the difference between n^2 and the whole number of cells in the cross — which is indeed evident. He further remarks in the first three cases below that these subsquares when extended with ‘part’ of the cross will display equal sums (he fails to note that they will not be found everywhere; see also above, p. 141).

In addition to the tables supplemented with the previous transcriptions, we have indicated how the cross has been filled (disregarding neutral placings).

28	75	70	29	40	61	20	83	78	21
74	25	32	71	62	39	82	17	24	79
31	72	73	26	63	38	23	80	81	18
69	30	27	76	37	64	77	22	19	84
41	59	43	57	50	53	36	66	67	33
60	42	58	44	48	51	65	35	34	68
4	99	94	5	49	52	12	91	86	13
98	1	8	95	54	47	90	9	16	87
7	96	97	2	46	55	15	88	89	10
93	6	3	100	56	45	85	14	11	92

<i>i</i>	·	<i>iii</i>	·	<i>x</i>	·
·	<i>ii</i>	·	<i>iv</i>	<i>viii</i>	·
				<i>ix</i>	·
				·	<i>vii</i>
				<i>vi</i>	·
				·	<i>v</i>

	4		2		5		
			1		3		
	7					6	

Commentary in $\mathcal{D}$. Square $n = 10$, with 4×4 subsquares and a separating cross.

ما	قس	لح	قنح	قنب	مب	فخ	قط	نط	قنب	نو	قم	قلد	س
قنا	ن	قنط	قند	نا	مو	قي	فز	قلج	سح	قلا	فكو	سط	سد
مد	قمح	مز	ند	قمه	قنح	قيا	فو	سب	قل	سه	عب	فكز	قله
مخ	نخ	قمو	قمز	مح	قند	فه	قيب	سا	عا	فكح	فكط	سو	قلو
قنز	قمح	نب	مط	قن	م	قب	صه	قلط	فكه	ع	نز	قلب	نح
قنه	لز	قنط	لط	مه	قنو	صز	ق	قلز	نه	قما	نز	سج	قلح
عز	قبط	قيح	ف	صا	قه	صح	قا	قز	فط	فا	قيه	قيد	فد
قك	عح	عط	قيز	قو	صب	صو	صط	ص	قح	قيو	فب	فخ	قيج
ه	قصو	ب	قصد	قنح	و	صد	قح	كج	قعه	ك	قعو	قع	كد
قفز	يد	قفه	قف	يه	ي	قد	صح	قسط	لب	قنز	قنب	لج	كح
ح	قند	يا	يخ	قفا	قنط	عو	قكا	كو	فسو	كط	لو	قسح	قعا
ز	يز	قنب	قنح	يب	قص	قكب	عه	كه	له	قسد	قسه	ل	قعب
قصح	قسط	يو	يخ	قفو	د	فكح	عد	قعه	قسا	لد	لا	قسح	كب
قضا	ا	قسه	ج	ط	قصب	عج	قكد	قعج	يط	قعر	كا	كر	قعد

(14 × 14) $\mathcal{D}$: 102^v.

لوح اربعة عشر متساوى الجمل كل صف من صفوفه ١٣٧٩. وفي اربع زوايا اربعة مربعات : *tit.*
كل واحد من ضرب ستة في ستة متساوية الجمل كل صف من صفوفها ٥٩١.

مشرح. ذو صليب ومربعات محلقة متوافقة اذا انضم اليه بعض الصليب مع احدها صار له وفق
مساوى للذى ينضم اليه بعضه معه ومجموع كل نظيرين من الصليب ١٩٧.

معج [(2,9)] *corr. ut vid. ex* سج.

عا	عح	عز	رمط	رن	رنا	رنب	عب	قمح	قعر	قح	قي	قط	ريز	ريخ	ريط	رك	قد
ع	لخ	رمو	ف	رمد	رلح	فد	رنه	قعر	قز	قب	قيه	ريد	قيب	ريب	رو	قيو	ركج
رنو	راز	صب	رله	رل	صيح	لخ	سط	قعر	قمو	ركد	ره	فكد	رج	قصح	فكه	فك	قا
سح	فو	رلد	فط	صو	رلا	رلط	رنز	قعه	قف	ق	قيح	رب	فكا	فكح	قسط	رز	ركه
ريخ	فه	صه	رلب	رلج	ص	رم	سز	قصح	قز	ركو	قيز	فكر	ر	را	فكب	رح	صط
رنط	رمج	ركط	صد	صا	رلو	فب	سو	قنح	قز	ركز	ريا	قصر	فكو	فكج	رد	قيد	صح
سه	رما	عط	رهم	فا	فز	رمب	رس	قسو	قنط	صز	رط	قيا	ريخ	قيح	قيط	ري	ركج
رنج	رمز	رمح	عو	عه	عد	عج	رند	قسا	قسد	ركا	ريه	ريو	قح	قز	قو	قه	ركب
قلز	قفز	قفو	قتم	قما	قنح	قفب	قمد	قشب	قسه	قسط	قنه	قعا	قنح	قنب	قعد	قعه	قنط
قنح	قلح	قلط	قفه	قند	قنب	قنح	قفا	قس	قسج	قنو	قع	قند	قعب	قعر	قنا	قن	قعو
ز	يد	يخ	شيد	شيه	شيو	ح	قلو	قفط	لط	مو	مه	رفا	رفب	رفج	رفد	م	
و	يط	شي	يو	شمع	شب	ك	شيط	قص	قله	لح	نا	رعح	مع	رعو	رع	نب	رفز
شك	شا	كح	رصط	رصد	كط	كد	ه	قصا	قلد	رفح	رسط	س	رسز	رสบ	سا	نو	لز
د	كب	رصح	كه	لب	رصه	شج	شكا	قلج	قصب	لو	ند	رسو	نز	سد	ريخ	رعا	رفط
شكب	كا	لا	رصو	رصز	كو	شد	ج	قلب	قصح	رص	نح	سج	رصد	رسه	نح	رعب	له
شكج	شز	رصح	ل	كز	ش	يخ	ب	قصد	قلا	رصا	رعه	رسا	سب	نط	ريخ	ن	لد
ا	شه	يه	شط	يز	كج	شو	شكد	قصه	قل	لج	رعج	مز	رعز	مط	نه	رعد	رصب
شيز	شيا	شيب	يب	يا	ي	ط	شبح	فكط	قصو	رفه	رعط	رف	مد	مج	مب	ما	رفو

(18 × 18) $\mathcal{D}$: 104^f.

لوح ثمانية عشر متساوي الجمل وحيلة كلّ صف من صفوفه ٢٩٢٥. وفي اربع زواياها اربعة *tit.*: مربعات محلقة كلّ مربع منها من ضرب ثمانية في ثمانية صفوف تلك المربعات من جميع جهاتها متساوية وحيلة كلّ صف منها ١٣٠٠. واذا انضم اليها بعض الصليب المحيط بها كانت متوافقة وكانت حيلة كلّ صف من صفوفها ١٦٢٥.

مشرح. ذو صليب ومربعات متساوية مجموع كلّ نظيرين من الصليب شكه.

١٣٤. (10, 4) قلب || رلو (6, 11) ريو || رب (6, 15) رب || ١٤٥ (10, 18) قمع || رب (5, 18) ريز

سد	رئج	رئج	سه	قح	رئط	رئد	قط	عب	رئج	ق	قفرز	قنب	قما	عو	رنا	رمو	عر
رئب	سا	رئط	رئج	قح	قئب	رئب	رئد	عا	قفو	قلاز	قئد	قئج	رن	عج	ف	رئز	
رئز	رس	رئسا	سب	قئا	رئو	رئز	قو	رئب	ع	قئد	قئف	قئح	عط	رئج	رئط	عد	
رئز	سو	رئج	رئد	رئج	قئ	قز	رك	سط	رئو	قئا	قئب	قئط	قئج	رئب	عه	رئب	
ق	ركز	ركب	قا	مح	رئط	رئد	مط	قئح	قئز	نو	رئا	رئو	نز	قئب	قئف	قئج	
ركو	صز	قد	رئج	رئج	مه	نئب	رئب	قئح	قئز	رئع	رئج	س	رئز	قئد	قئط	قئا	
قئج	ركد	ركه	صئج	نا	رئو	رئز	مو	قئو	قئط	نط	رئج	رئط	ند	قئف	قئب	قئج	
ركا	قئب	صئط	رئج	رئج	ن	مز	رف	قئسا	قئد	رئب	نئج	نه	رئب	قئط	قئد	قئا	
م	رفو	رفز	لئز	قئب	قئد	قئف	قئط	قئب	قئف	قئط	قئف	قئا	قئج	مد	رئب	رئج	
رفه	لئط	لئح	رئج	قئج	قئا	قئ	قئو	قئس	قئج	قئو	قئد	قئب	رئا	مئج	مئب	رئد	
صئب	رئله	رئل	صئج	لئب	رئف	رئص	لئج	قئح	قئز	كد	شئج	رئصئ	كه	قئد	رئج	قئف	
رئد	قئط	صئو	رئلا	رئد	كئط	لئو	رئسا	قئح	قئز	شئب	كئح	رئصئ	رئب	قئا	قئح	قئط	
صئه	رئلب	رئلئج	صئ	لئه	رئب	رئصئ	لئ	قئط	قئو	كئز	شئ	كئب	قئز	رئ	رئا	قئب	
ركئط	صد	صئا	رئلو	رئط	لئد	لئا	رئو	قئف	رئصئ	كو	كئج	شئد	قئز	قئو	قئج	رئد	
رئو	شئبا	شئو	رئز	قئد	رئج	رئلئح	قئف	قئب	شئج	قئو	رئبا	قئز	دئ	شئج	شئج	هئ	
شئى	رئج	كئ	شئز	رئب	قئا	قئح	رئط	شئب	رئى	قئج	قئك	رئز	كئب	ائ	حئ	شئط	
رئط	شئج	شئط	رئد	قئز	رئم	رئما	قئب	شئب	قئط	رئح	رئط	قئد	زئ	شئك	شئكا	بئ	
شئه	رئج	رئب	شئب	رئز	قئو	قئح	رئد	طئ	شئو	رئه	قئج	قئب	رئب	شئز	وئ	جئ	

(18 × 18) D : 105^r.

لوح ثمانية عشر متفق الجمل كل صف منه ٢٩٢٥. والمربع الذى فى وسطه من ضرب عشرة : tit. فى عشرة متفق الجمل ايضاً كل صف من صفوفه ١٦٢٥. وفى اربع زواياه اربعة مربعات كل واحد منها ثمانية فى ثمانية متفقة الجمل كل صف منها ١٣٠٠. وكل واحد من هذه الاربعة المربعات مقسوم باربعة مربعات كل واحد منها من ضرب اربعة (فى اربعة) متفقة الجمل كل صف منها ٦٥٠. ومن هذه المربعات اربعة على القطرين كل واحد منها مشارك فى العشرة التى فى الوسط وثمانية متصلة بالمربعات التى فى الزوايا.

مشرح. ذو صليب وفواصل ومربعات متساوية مجموع كل نظيرين من الفاصلة ٣٢٥.

(8,9) قئو || ut vid. ١٠٣ (16,13) قد || رئا (11,13) رئه || ١٥١ (10,13) قئح || ١٠٦ (11,18) قط
١٢٣ (1,6) قئب || (قئح) = ١٨٨ (10,7) قئح || (قئو sc.) ١٢٦ corr. ex ١٢٤ (4,8) قئد || ١٧٦
ut vid. || ٣ pr. scr. (v. seq.). (13,1) قئو || ١٧١ (10,6) قئط ||

64	263	258	65	108	219	214	109	72	253	140	187	182	141	76	251	246	77
262	61	68	259	218	105	112	215	254	71	186	137	144	183	250	73	80	247
67	260	261	62	111	216	217	106	255	70	143	184	185	138	79	248	249	74
257	66	63	264	213	110	107	220	69	256	181	142	139	188	245	78	75	252
100	227	222	101	48	279	274	49	168	157	56	271	266	57	132	195	190	133
226	97	104	223	278	45	52	275	158	167	270	53	60	267	194	129	136	191
103	224	225	98	51	276	277	46	166	159	59	268	269	54	135	192	193	130
221	102	99	228	273	50	47	280	161	164	265	58	55	272	189	134	131	196
40	286	287	37	152	174	175	149	162	165	169	155	171	153	44	282	283	41
285	39	38	288	173	151	150	176	160	163	156	170	154	172	281	43	42	284
92	235	230	93	32	295	290	33	148	177	24	303	298	25	124	203	198	125
234	89	96	231	294	29	36	291	178	147	302	21	28	299	202	121	128	199
95	232	233	90	35	292	293	30	179	146	27	300	301	22	127	200	201	122
229	94	91	236	289	34	31	296	145	180	297	26	23	304	197	126	123	204
16	311	306	17	84	243	238	85	12	313	116	211	206	117	4	323	318	5
310	13	20	307	242	81	88	239	314	11	210	113	120	207	322	1	8	319
19	308	309	14	87	240	241	82	315	10	119	208	209	114	7	320	321	2
305	18	15	312	237	86	83	244	9	316	205	118	115	212	317	6	3	324

	14		8		13		4							12			
	9		16				15							5			
	18		2			1										17	
	10		19		3		20							6			
	21		11		22		7							23			

·	v				
vi	·				
·	vii				
ix	·				
x	·	·	iii	·	i
viii	·	iv	·	ii	·

Commentary in D. Square $n = 18$, with 4×4 (or 8×8) subsquares in the corners and a cross. As noted by the commentator, the inner 10×10 square is also magic.

IV. General commentary

Introduction

(§ 0) Translator's introduction

The translator, who had previously no knowledge of magic squares, tells us in detail how he came to know them. First, he says, there was the 3×3 square mentioned (or 'reported', *dhakara*) by Nicomachos. Now there is no mention of a magic square in the *Introduction to arithmetic*, nor in its Arabic translation by Thābit ibn Qurra (the translator's father?). But a 3×3 square may have appeared as a marginal addition. Still, the translator's allusion is problematic.

All other scholars mentioned in this introduction belong to the ninth century; they are known from contemporary writings or also from the *Fihrist* of Ibn al-Nadīm, written in the end of the tenth century, which in particular lists contemporary Arabic books or translations.

The first scholar mentioned is Abū'l-Qāsim al-Ḥijāzī, of the first half of the ninth century, who saw a 4×4 square with the first sixteen natural numbers and is said to have been impressed by its realization with these numbers. A mathematician would have admired the arrangement rather than the use of a continuous number sequence; but then al-Ḥijāzī was a theologian.

Ishāq ibn Ḥunayn (d. 910/1), a contemporary of Thābit ibn Qurra, translated among others Euclid's *Elements*, a translation then revised by Thābit himself. According to our author, Ishāq represented a 6×6 square on the 'back' of a copy of Euclid's book.

In an unspecified book our translator saw a few magic squares, all of orders less than ten. Knowing that already in early Islamic times the first seven squares were attributed to the (then) known planets,¹⁹⁵ one wonders if that was not also the case in ancient writings.

The two exemplars used by our translator for his work were found, he says, 'in the Library, among the books of the caliphs' collection'. Now this library must be that of the *Bayt al-ḥikma*, the 'House of wisdom',

¹⁹⁵ Above, pp. 6–7; see also *Magic squares*, pp. 11–12, 16–19 (and earlier editions: *Les carrés magiques*, pp. 10, 14, 253–266; *Маг. квадраты*, pp. 18, 23, 263–272).

indeed known to have preserved works collected by the early caliphs.¹⁹⁶ Since there were two exemplars, the translator could complete his work, using the parts preserved in one to make up for the parts of the other destroyed by termites.¹⁹⁷

But our translator was not the first to have paid attention to this treatise. As he reports, there was a note by (Muḥammad b. ʿĪsā b. Aḥmad) al-Māhānī, who was a Persian mathematician and astronomer working in Baghdad in the middle of the ninth century, known to have commented various Greek works; it is therefore hardly surprising to see him examining the two manuscripts of our treatise. His note was not the only one: more extensive was that by Ḥasan (as commonly) b. Mūsā al-Nawbakhtī, a theologian and philosopher of the second half of the ninth century, who is said by Ibn al-Nadīm to have been in relation with translators of (Greek) philosophical works.¹⁹⁸

(§1) Definition of a magic square

The first leaf of a manuscript is generally that which is less well preserved. This may explain why the definition of a magic square (§1) is somewhat imprecise, in particular without any mention of the constant sum being also found in the two main diagonals.

(§2) Two ways of constructing magic squares

The next paragraph tells us about the two main kinds of magic squares, namely ordinary ones and bordered ones, and the difference between the ways in which they are constructed. But there is no allusion to their main difference, that is, that all the inner squares of a bordered square are also magic, despite such characteristic being essential to the subject treated.

As for the construction of an ordinary magic square, it is based on the *natural square*, that is, a square of the same order as the one to be constructed but containing the first numbers in their natural order. Such a square has two notable properties: first, its main diagonals already contain the magic sum; second, as implied in our text, pairs of symmetrically placed lines or columns display differences with the magic sum which have

¹⁹⁶ Eche, *Bibliothèques*, pp. 26–27, 36, 56–57.

¹⁹⁷ Ibn al-Nadīm also mentions the damage done by termites to manuscripts (*Fihrist*, ed. Flügel *et al.*, I, 243, 27; tr. Dodge, II, p. 585). The impressive result of such damage to a book, after termites first attacked its book case, is shown in T. E. Snyder's study, p. 114 (however, our two MSS. cannot have been in such a state); see also Ghidini's article. Some known Italian libraries suffered likewise, both in the south (Palermo, Catania) and in the north (Florence, Venice).

¹⁹⁸ *Fihrist*, I, 177, 11–12; tr. Dodge, I, 441.

the same amount but with a different sign. To obtain a magic square, one would then normally, without modifying the diagonals, proceed to exchange numbers between symmetrically placed lines and columns until reaching the magic sum.

Having stated that constructing a magic square from exchanges in the natural square of the same order ‘is a method presenting difficulty for the beginner’, our text turns to the ‘different, and easier, way’ for the construction of bordered squares. This is the only allusion of our text to ordinary magic squares. It at least indicates that ordinary magic squares were also studied in Greece.

Note on the construction of ordinary magic squares

As the study of Būzjānī’s treatise, the earliest originally Arabic one on magic squares (Antākī’s is in this respect irrelevant), shows, his exchanges between opposite rows of the natural square turned out initially to be successful only for small orders; that is, they did not give rise to general methods. Būzjānī indeed considers exchanging numbers in natural squares of odd order first for the (banal) case of 3, then for the order 5; and the two ways he then suggests for the latter are particular to that order and cannot be extended to higher orders. The exchanges for even orders turn out to be easier, but here again Būzjānī did not reach a general method after considering the first even orders.¹⁹⁹ For, to reach a general method, one has to consider exchanges in the natural square as a whole, and not by considering one order after the other. That is why general methods were not found before the 11th century, when individual treatments were abandoned and general properties of the natural square set out.²⁰⁰

Part I: Construction of odd-order bordered squares

(§§ 3–6) The method found in our treatise

We shall now thoroughly examine the method our text presents; first, explaining the general placing to be memorized; then how this placing must have been found empirically; and ending with the mathematical foundation of this general placing.

¹⁹⁹ See *Magic squares in the tenth century*, pp. 30–36.

²⁰⁰ See above, p. 5 and n. 3.

1. Description of this method

This method may be generally described, when one (unlike our text) takes the numbers from the smallest on, here 1, and puts them in the outer border first, as follows. Starting next to a corner cell, we put in the rows meeting there the natural sequence, alternatively in one and then the other, till we reach the middle cells. We write the next number in the cell next to the cell just reached (7 in the figure), then move to the opposite corner (that of the row where we started), then put the next in the middle cell on the other side, the next in the corner of the same row (just one possibility since the other corner is for a complement), whereupon we resume placing on either side of this corner, but this time *towards* the corner cell. Opposite cells are each time left empty for the complements. A characteristics of this method is to separate the numbers by parity: the vertical rows contain, except for the corner cells, only odd numbers, while in the horizontal rows are found, except for their middle cells, even numbers. The subsequent mathematical justification will provide the reason for this anomaly.

8					12	14	16	10
								15
								13
								11
								9
5								
3								
1								
	2	4	6	7				

Fig. 6

Since our treatise gives the above general method but does not explain its foundation, we shall first consider how it must have been obtained and, second, provide its mathematical justification.

2. Discovery of this general method

Since the construction of bordered squares is described in our treatise, but not justified, Būzjānī attempted to trace back the origin of this method. In the present case of odd-order squares, he succeeded. So let

us follow, in our writing, his path towards the empirical finding of this method.²⁰¹

The outer border of a square of order m comprises $4m - 4$ cells, to be filled with $2m - 2$ pairs of complements. Consider then, as Būzjānī does, a square of order $m = 5$, with 16 cells in its border, thus to be occupied by the numbers $1, \dots, 8$ and $18, \dots, 25$. We are first to fill the central square of order 3. To do so, we start by placing in the central cell the median number, $\frac{m^2+1}{2} = 13$, and then take the $2 \cdot 3 - 2 = 4$ numbers on each side of 13 and arrange them in pairs of complements,

$$\begin{array}{cccc} 9 & 10 & 11 & 12 \\ 17 & 16 & 15 & 14, \end{array}$$

which we write in the border surrounding the central cell in the known way (Fig. 7–8).²⁰²

2	9	4
7	5	3
6	1	8

Fig. 7

	10	17	12	
	15	13	11	
	14	9	16	

Fig. 8

The sum of opposite cells is thus 26, and, as seen (p. 2), this is the sum we are also to find in opposite cells of the outer border of order 5, namely using the pairs

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18. \end{array}$$

The first placing to be considered is that of the corners, for it will determine the filling of two rows. Būzjānī chooses to put in the two upper corner cells the smaller numbers 4 and 6, with their complements opposite (Fig. 9). But he also notes that there would be other possibilities—though not many for this case of the 5×5 square. As he asserts, the only possible pairs of smaller numbers for two consecutive corner cells with the first numbers are 1, 3; 3, 7; 5, 7; 2, 8; 4, 6; 6, 8, which would give altogether ten borders.²⁰³ His choice is obviously made in conformity

²⁰¹ See the edition of his treatise, pp. 150–157 and (Arabic) 209–216; or *Magic squares in the tenth century*, B.15–16.

²⁰² Putting 2 in the upper left-hand corner cell and 4 in the right-hand one fits the general method to be explained.

²⁰³ Given by us in *Magic squares in the tenth century*, pp. 44–46 and *Magic squares*, p. 144 (*Les carrés magiques*, pp. 120–121; *Маг. квадраты*, p. 131).

with the Greek method as, too, with the case of the 3×3 square, with 2 and 4: the upper corner cells of the outer border of a square of odd order n , filled with the first small numbers and their complements, will be $n - 1$ and $n + 1$. Accordingly, for an inner border of order m , they will be the $(m - 1)$ th and the $(m + 1)$ th of the small numbers which are to occupy this border.

Būzjānī is then left with completing the border, or, rather, two contiguous rows of it since complements will occupy the cells opposite. Their choice must complete the still required sum. But their arrangement, he writes, is irrelevant; thus the configuration chosen should be convenient for memorization. He then completes the required sum for this order, 65, first horizontally (Fig. 10), then vertically (Fig. 11), and puts the complements opposite. Here again, his choice is to fit the general arrangement he has in mind.

4				6
	10	17	12	
	15	13	11	
	14	9	16	
20				22

Fig. 9

4	24	23	8	6
	10	17	12	
	15	13	11	
	14	9	16	
20	2	3	18	22

Fig. 10

4	24	23	8	6
19	10	17	12	7
21	15	13	11	5
1	14	9	16	25
20	2	3	18	22

Fig. 11

To construct the square of order 9, Būzjānī proceeds in the same manner. He first takes the median number, 41, which he puts in the central cell (Fig. 12). Then he arranges the four numbers preceding it and the four numbers following it in the border of order 3. To the border of order 5 he allots the eight numbers from 29 to 36 and from 46 to 53, placed in the same way as were the corresponding ones in the border already constructed (Fig. 11). He then considers the two sequences of twelve numbers for the border of order 7,

17 18 19 20 21 22 23 24 25 26 27 28
65 64 63 62 61 60 59 58 57 56 55 54.

He begins by writing 22 and 24 in the corners, which is in accordance with his previous choices: he takes here the sixth and the eighth smaller numbers. He is left with finding five numbers making, together with 22 and 24, 287, which is the sum found in the main diagonals (and thus the required magic sum for the 7×7 inner square); with the remaining numbers, he completes the vertical rows. There remain the numbers from 1 to 16 and 66 to 81 for the border of order 9, the eighth and tenth

of the first sequence being for the corner cells, and the others used to complete, by trial and error, the sums in the border's horizontal and vertical rows.

8	80	78	76	75	12	14	16	10
67	22	64	62	61	26	28	24	15
69	55	32	52	51	36	34	27	13
71	57	47	38	45	40	35	25	11
73	59	49	43	41	39	33	23	9
5	19	29	42	37	44	53	63	77
3	17	48	30	31	46	50	65	79
1	58	18	20	21	56	54	60	81
72	2	4	6	7	70	68	66	74

Fig. 12

This work being done, Būzjānī infers from the arrangement found, as he writes, *a method to improve the student's skill and (also) intended for those who prefer to save themselves the trouble of working out which numbers to arrange in the square.*²⁰⁴ This is the general way of constructing odd-order bordered squares found in the Greek treatise.

3. Mathematical basis of this method

The principle for constructing a border, of whatever order, is as follows. Consider we have a magic square of order $n - 2$, in which are arranged the $(n - 2)^2$ first natural numbers. Its magic sum is therefore

$$M_{n-2} = \frac{n-2}{2} [(n-2)^2 + 1].$$

We now wish to surround it with a border of order n . Since this border comprises $2n + 2(n - 2) = 4n - 4$ cells, it has to be filled with $2n - 2$ small numbers and their complements to $n^2 + 1$. As usual, we shall consider putting in this outer border the $2n - 2$ first numbers and their complements. So we shall, to begin with, uniformly increase each number of the square already constructed by $2n - 2$. The sum in any row of this square (including the two main diagonals), being increased by $n - 2$ times this quantity, becomes

$$M'_{n-2} = \frac{n-2}{2} [(n-2)^2 + 1 + 2(2n-2)] = \frac{n-2}{2} [n^2 + 1],$$

²⁰⁴ Edition, pp. 157–158 & 216–217; *Magic squares in the tenth century*, B.17; *Magic squares*, pp. 146–147 (*Les carrés magiques*, pp. 121–122; *Маг. квадраты*, pp. 132–133).

with the smallest number in the square now being $2n - 1$ and the largest, $(n - 2)^2 + 2n - 2 = n^2 - 2n + 2$.

We are now to fill the $4n - 4$ cells of the new border with the two sequences

$$1, 2, \dots, 2n - 2$$

$$n^2 - 2n + 3, n^2 - 2n + 4, \dots, n^2.$$

Since the magic sum must be increased by

$$M_n - M'_{n-2} = n^2 + 1,$$

we shall arrange in pairs of complements the numbers to be placed, that is, we shall superpose the above two sequences with the second taken in reverse order:

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & 2n - 2 \\ n^2 & n^2 - 1 & n^2 - 2 & \dots & n^2 - 2n + 3, \end{array}$$

and put these pairs at either end of each horizontal row, column and main diagonal. We are now left with the problem of obtaining the magic sum in each of the four border rows.

Let us apply that to our case of odd orders. Since the outer border of our square of order $n = 2k + 1$ contains $4n - 4 = 8k$ cells, it will be filled with the following four sequences of k smaller numbers

$$\sum_1^k a_g, \quad \sum_1^k b_h, \quad \sum_1^k c_i, \quad \sum_1^k d_j,$$

and the corresponding four sequences of larger numbers, namely their complements. Note too that, since two of the smaller numbers are to occupy two consecutive corner cells, and therefore their complements the two opposite corners, each of two consecutive rows will contain $k + 1$ smaller numbers and k complements, whereas the other two will accordingly contain k smaller numbers and $k + 1$ complements.

Consider now taking the smaller numbers in natural order. These four sequences of smaller numbers will have the form

$$\begin{array}{llll} \{a_g\} & 1, 2, \dots, k & \text{with sum} & \frac{k(k+1)}{2} \\ \{b_h\} & k + 1, k + 2, \dots, 2k & \text{with sum} & k^2 + \frac{k(k+1)}{2} \\ \{c_i\} & 2k + 1, 2k + 2, \dots, 3k & \text{with sum} & 2k^2 + \frac{k(k+1)}{2} \\ \{d_j\} & 3k + 1, 3k + 2, \dots, 4k & \text{with sum} & 3k^2 + \frac{k(k+1)}{2}. \end{array}$$

We could choose to consider placing these sequences as they are and modify the places of some of their elements to arrive at the required

$$\begin{aligned}
&= k^2 - 2k + 1 + A + (k+1)(n^2 + 1) - 3k^2 + 2k + 1 - C - D \\
&= A - C - D - 2k^2 + 2 + (k+1)(n^2 + 1).
\end{aligned}$$

Since this must equal $M_n = (k + \frac{1}{2})(n^2 + 1)$, we shall have

$$C + D - A + 2k^2 - 2 = 2k^2 + 2k + 1$$

and therefore

$$C + D - A = 2k + 3. \quad (*)$$

On the other hand, the sum in the upper row IV will be

$$\begin{aligned}
&\sum d_j + A + D + (k-1)(n^2 + 1) - \sum b_h + (n^2 + 1) - B \\
&= A + D - B + 3k^2 - k - 2 - k^2 + k + k(n^2 + 1) \\
&= A + D - B + 2k^2 - 2 + k(n^2 + 1),
\end{aligned}$$

and we shall therefore have

$$A + D - B + 2k^2 - 2 = 2k^2 + 2k + 1,$$

whence

$$A + D - B = 2k + 3. \quad (**)$$

Now using the four still unplaced numbers, $2k-1$, $2k$, $2k+1$, $2k+2$, we have the two combinations

$$(2k+2) + (2k+1) - 2k = 2k+3$$

$$(2k+2) + 2k - (2k-1) = 2k+3.$$

Comparing the terms in these expressions with our two relations $(*)$ and $(**)$ we shall take $A = 2k$, $B = 2k-1$, $C = 2k+1$, $D = 2k+2$.

10	120	118	116	114	113	14	16	18	20	12
103	28	100	98	96	95	32	34	36	30	19
105	87	42	84	82	81	46	48	44	35	17
107	89	75	52	72	71	56	54	47	33	15
109	91	77	67	58	65	60	55	45	31	13
111	93	79	69	63	61	59	53	43	29	11
7	25	39	49	62	57	64	73	83	97	115
5	23	37	68	50	51	66	70	85	99	117
3	21	78	38	40	41	76	74	80	101	119
1	92	22	24	26	27	90	88	86	94	121
110	2	4	6	8	9	108	106	104	102	112

Fig. 14

6	48	46	45	10	12	8
39	16	36	35	20	18	11
41	31	22	29	24	19	9
43	33	27	25	23	17	7
3	13	26	21	28	37	47
1	32	14	15	30	34	49
42	2	4	5	40	38	44

Fig. 15

This thus gives the placing method of our treatise and shows, incidentally, why four small numbers, in the upper corners and middle cells, break the otherwise regular succession, either of numbers or of parity (see in Fig. 14, with $k = 5$, the places of 10, 9, 11, 12; the place of 11 explains why the numbers following 12 were placed towards the corner cell, see p. 166).

We have seen the cases $n = 3$ (Fig. 7), $n = 5$ (Fig. 11) and $n = 9$ (Fig. 12); our treatise also has the two cases $n = 11$ and $n = 7$ (Fig. 14–15).

As already said (p. 13), this section on the construction of odd-order magic squares is omitted in MS. *D*; for, we are told, it is wordy and may be replaced by a single table, that set out by *Khāzinī*, who starts with the numbers in each border afresh.²⁰⁷ With the numerals adapted to our writing, this table corresponds to Fig. 16: the Roman numerals indicate the sequence of smaller numbers for filling each border, $\circ$ stands for $\frac{n^2+1}{2}$, and dots for complements.

<i>vi</i>	.	.	.	<i>x</i>	<i>xii</i>	<i>viii</i>
.	<i>iv</i>	.	.	<i>viii</i>	<i>vi</i>	<i>xi</i>
.	.	<i>ii</i>	.	<i>iv</i>	<i>vii</i>	<i>ix</i>
.	.	.	$\circ$	<i>iii</i>	<i>v</i>	<i>vii</i>
<i>iii</i>	<i>i</i>	.	<i>i</i>	.	.	.
<i>i</i>	.	<i>ii</i>	<i>iii</i>	.	.	.
.	<i>ii</i>	<i>iv</i>	<i>v</i>	.	.	.

Fig. 16

²⁰⁷ See MS. *D* fol. 60^v, 11 (or p. 16 *n*) for the Arabic text and fol. 61^r for the figure.

Part II: Odd-order bordered squares with separation by parity

We now come to the longest (§§ 7–35) and most complicated construction of the treatise, that of bordered squares displaying separation by parity, the odd numbers all being located in a rhomb (rather, an oblique square) within the main square. Because of its difficulty, this construction left few traces later. Thus, an eleventh-century author, speaking on the subject of separation by parity for odd-order squares, says that he left out ‘difficult methods’ as being beyond the scope of his work. For there are, he adds, various simpler methods. But he omits to say that the difficulty is only for bordered squares.²⁰⁸

Note on this separation for ordinary magic squares

In the case of ordinary magic squares, the construction is very simple, and one way to obtain such a configuration is the following.²⁰⁹ We take an empty square of the considered order and draw in it an oblique square the corners of which will bisect each side. With the parallels drawn through the points of intersection of that square’s sides with the main square’s rows, there appears a square of the same order as the original one. After filling it with the natural sequence of numbers (Fig. 17), we observe that some numbers, the odd ones, fall in cells of the main square while others, the even ones, are on the crossings. Let us now move the latter ones, divided into four groups, into the empty corner triangles opposite (Fig. 18). We thus obtain a magic square with the desired separation (Fig. 19): the even numbers are in the corners whereas the others, which have stayed in place, are within the rhomb.

		1		
	6	2		
11	7	3		
16	12	8	4	
21	17	13	9	5
22	18	14	10	
23	19	15		
24	20			
	25			

Fig. 17

14	10	1		
	6	2		
20	11	7	3	
16	12	8	4	
21	17	13	9	5
22	18	14	10	
23	19	15		
24	20			
	25			

Fig. 18

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

Fig. 19

²⁰⁸ See our *Un traité médiéval*, pp. 36 & 201.

²⁰⁹ See *Magic squares*, pp. 34–36 (*Les carrés magiques*, pp. 33–35; *Маг. квадраты*, pp. 42–44).

(§ 7) The main square and its parts

2	9	4
7	5	3
6	1	8

Fig. 20

For the construction of bordered squares displaying this separation, we are first to consider the different parts of the main square. As an example, we can consider the first case, that of order 3, which *naturally* displays the required arrangement, and this may have given rise to the idea of extending it to larger orders. As seen (Fig. 20), the main square contains an oblique square, which itself includes a smaller square (in this case the central cell). And this situation occurs in all larger squares (Fig. 21), and displays some peculiarities which will turn out to be essential for the construction.²¹⁰

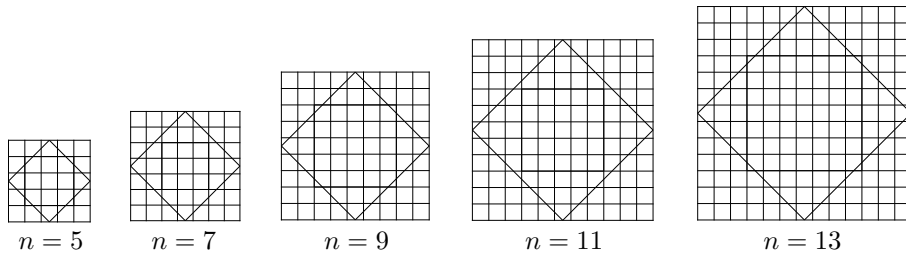


Fig. 21

The main square, of odd order $n = 2k + 1$, contains $n^2 = 4k^2 + 4k + 1$ cells. The inner oblique square will contain as many cells as there are odd numbers, thus $2k^2 + 2k + 1$, and the quantity of $2k^2 + 2k$ even numbers will be equally divided among the outer triangles, thus $\frac{1}{2}k(k+1)$ elements in each — an integer, since the product of k by $k+1$ is even. Now, of these $2k^2 + 2k + 1$ odd numbers, a part will be contained in the largest possible square within the rhomb while the remainder will occupy the equal triangles completing the rhomb. From Fig. 21 and the table below it appears that the inner square is the same for two consecutive odd orders $4t + 1$ and $4t + 3$, namely

²¹⁰ This is one of the parts amply commented by Būzjānī; see *Magic squares*, pp. 177–182 (*Les carrés magiques*, pp. 153–157; *Маг. квадраты*, pp. 163–168); or also *Magic squares in the tenth century*, B.21, i-ii, or ll. 757–784 in the edition of the Arabic text. But there is no contribution of his to the later, more difficult parts — rather the opposite since he turns to an empiric way of filling (*ibid.*).

$$\frac{(4t+1) + (4t+3)}{4} = 2t+1.$$

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$n = 4t + 1$	5	9	13	17	21
$n = 4t + 3$	7	11	15	19	23
$2t + 1$	3	5	7	9	11

Fig. 22

On the other hand, the second table below shows that the four triangles will contain the same number of elements for two consecutive odd orders, but just not the same order pairs as before; indeed, each of the four triangles will contain t^2 cells for the two orders $4t \mp 1$. The above Fig. 21 shows how the same triangles can be found in two different (consecutive) squares.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$n = 4t - 1$	3	7	11	15	19
$n = 4t + 1$	5	9	13	17	21
$4t^2$	$4 \cdot 1$	$4 \cdot 4$	$4 \cdot 9$	$4 \cdot 16$	$4 \cdot 25$

Fig. 23

Summarizing, of three consecutive odd orders $4t+1$, $4t+3$, $4t+5$, the oblique squares of the first two contain on the one hand (largest) inner squares of equal size, and, on the other hand, triangular parts surrounding these equal inner squares which are unequal since these squares are surrounded by, respectively, t and $t+1$ borders. For the last two orders, the largest inner squares will be of unequal size, with sides $2t+1$ and $2(t+1)+1$ respectively, while the triangular parts of the oblique square will be equal since both inner squares are surrounded by $t+1$ borders.

Remark. As we shall see, arranging the odd numbers supposes knowledge of the above method for filling odd-order bordered squares, both for filling the inner squares and the triangular parts of the oblique square. Furthermore, for placing the even numbers, a specific preliminary arrangement is followed by straightforward placings of consecutive numbers, just as in the construction of even-order bordered squares (below, pp. 206–207). This indicates intimate knowledge in Greek antiquity of how to construct common bordered squares.

A. Placing the odd numbers

(§8) Filling the inner square

Placing the numbers in the inner square is easy: we shall put $\frac{n^2+1}{2}$ (odd) in the central cell, then fill the borders successively with the subsequent smaller and larger numbers, just as we did with bordered squares, but here taking only odd numbers. The largest inner square will thus be occupied by two sequences of consecutive odd numbers on either side of $\frac{n^2+1}{2}$, and it will display its sum due as well as the characteristics of a bordered square.

This is exemplified in Fig. 24, of order $n = 11$ (see above, Fig. 11 or 16). The smallest term reached, and last placed (in the outer border since the text begins with filling the centre) will be 37.

Remark. If we wish to start, as we did with common bordered squares, in the outer border of this inner square and place ascending sequences of smaller numbers (that border being at least of order 5), we shall put the smallest number above the lower left-hand corner cell; this number will be $4t^2 + 1$ if n (≤ 9) has the form $4t + 1$, but $4(t + 1)^2 + 1$ for the form $4t + 3$. Next we shall put the subsequent smaller odd numbers alternately around the corner, then in the middle and corner cells, finally alternately on both sides of the opposite corner, just as we did for usual bordered odd-order squares.

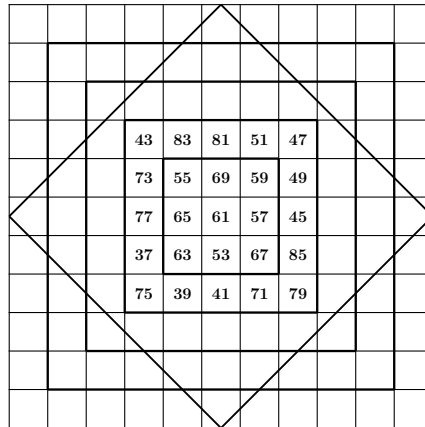
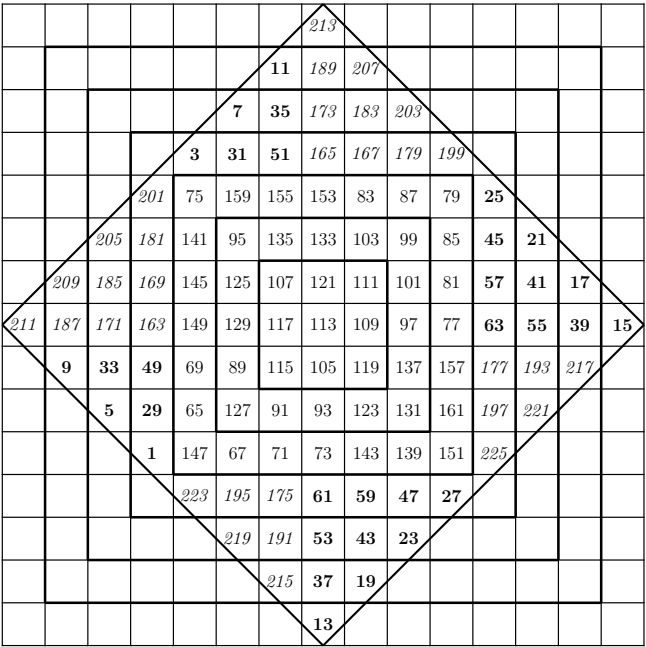


Fig. 24

The inner square being thus a bordered square filled with two continuous sequences, we may proceed with placing the next odd numbers in the triangular parts.

(§§ 9–11) Filling the remainder of the rhomb

The way to place the remaining odd numbers is clearly seen from the text. After filling the inner square with the median number and the next numbers on either side of it, the subsequent odd numbers are placed in the successive borders of the main square, but *within the rhomb*, beginning with the border closest to the inner square. We shall thus have filled the larger halves of the triangles on the right (left in the Arabic text) and below, and the smaller halves of the triangles opposite, namely the parts left empty after placing the complements of the numbers placed first (Fig. 25). This is done progressing from inside to out with the decreasing sequence of the smaller numbers.



[illegible]

Fig. 26

Seen that way, this placing indeed recalls the method taught earlier for odd-order bordered squares, with its alternate movement placing the successive numbers in two contiguous rows (Fig. 6, p. 166), and might have been inferred from it. This, though, does not appear from the extant text, which merely states which odd number will occupy which border.

these borders being taken in turn from inside; nevertheless, the general placing will be inferred from it, just as for common bordered squares the general placing was inferred from successive examples.

(§ 12) Different treatments for placing the even numbers

The treatment for all types of (odd) orders has so far been the same. But it now becomes necessary to differentiate between the two order types $n = 4t + 1$ and $n = 4t + 3$. Indeed, consider the cells remaining empty in each of the successive borders' rows of the main square (including the corner cells, common to two rows); their number is a multiple of 4 in the first case and an evenly-odd number in the second: see the table below, with p numbering the incomplete borders from the smallest, thus the one surrounding the square already filled (or above, Fig. 21). Thus the procedure cannot be the same if we wish to be left, after the equalization, with straightforward placings, as said above (p. 176). We shall therefore consider the orders $n = 4t + 1$ with $t \geq 2$ (since $n = 5$ is a particular case) and $n = 4t + 3$ separately.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
$n = 4t + 1$	4	8	12	16	20
$n = 4t + 3$	2	6	10	14	18

Fig. 27

Our text next considers the situation in each border for the order $n = 4t + 1$, beginning with the smallest.

(§ 13) Quantity of empty border cells in squares of order $n = 4t + 1$

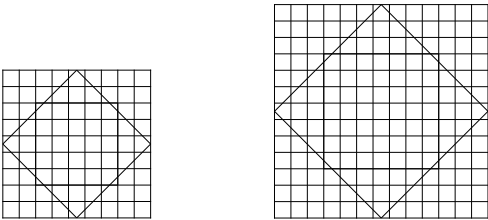


Fig. 28

As known from § 7, if the main square has the order $n = 4t + 1$, the inner square will have the order $2t + 1$. That is, there will be a quantity of t borders partly filled with odd numbers; from the informations given in the text we infer that the first six will leave the following number of empty cells N_p :

N_1	N_2	N_3	N_4	N_5	N_6
12	28	44	60	76	92

Fig. 29

Generalizing, and having in mind the previous results (§ 12), we may say that

In a square of order $4t + 1$, with an inner square of order $2t + 1$ completely filled and thus with t borders partly filled, the p th border starting from the inner square contains $16p - 4$ empty cells, with $4p$ in each of its rows ($p = 1, \dots, t$).

(§ 14–15) Excesses and deficits in each row for the order $n = 4t + 1$

Since the magic sum for a square of order n is $M_n = n \cdot \frac{n^2+1}{2}$, the sum due for m cells in one row in order to satisfy the magic condition is $m \cdot \frac{n^2+1}{2}$ (above, p. 2). Let us examine the situation in each row after the previous placing of odd numbers —disregarding the inner square since each of its rows makes the sum due.

Relative to the cells already filled, the upper rows in each incomplete border display an excess over the sum due, as do also the (for us) left-hand columns, while the opposite rows show a deficit of equal amount since they are filled with complements. Our text gives all necessary information about the excesses or deficits for the orders $n = 4t + 1$, starting from the border surrounding the inner square — in fact, in the absence of symbolism, it merely gives the values for the first orders and notes the way they increase, but this is sufficient to set out and extend at will a table of the differences Δ displayed by the horizontal rows (lines, L) and the vertical ones (columns, C) according to both the value of t for the order $n = 4t + 1$ (we include $n = 5$) and the number of the border:

$n = 4t + 1$	Δ_1^L	Δ_1^C	Δ_2^L	Δ_2^C	Δ_3^L	Δ_3^C	Δ_4^L	Δ_4^C	Δ_5^L	Δ_5^C
$t = 1$	12	10								
$t = 2$	20	18	36	34						
$t = 3$	28	26	52	50	76	74				
$t = 4$	36	34	68	66	100	98	132	130		
$t = 5$	44	42	84	82	124	122	164	162	204	202

Fig. 30

These data are sufficient for us to express the differences in a formula. Let us designate by $\Delta_p^{L, 4t+1}$ the excess of the upper row belonging to the p th border if the order of the main square is $n = 4t + 1$, and by $\Delta_p^{C, 4t+1}$ that of the left-hand row of the same border. We infer from the

table that $\Delta_1^{L,4t+1} = 8t + 4$, $\Delta_2^{L,4t+1} = 16t + 4$, $\Delta_3^{L,4t+1} = 24t + 4$, $\Delta_4^{L,4t+1} = 32t + 4$, thus, generally,

$$\Delta_p^{L,4t+1} = 8pt + 4 = 2p(n-1) + 4.$$

As for the lateral rows of these borders, their differences happen to be less than those of the horizontal rows by 2, and we shall thus have

$$\Delta_p^{C,4t+1} = 8pt + 2 = 2p(n-1) + 2.$$

(§ 16) Quantity of empty border cells in squares of order $n = 4t + 3$

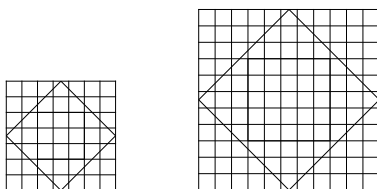


Fig. 31

The number N_p of empty cells in each border p cut by the sides of the oblique square, starting with the innermost ($p = 1$), is as indicated in the table below, which merely extends the data of the text.

N_1	N_2	N_3	N_4	N_5	N_6
4	20	36	52	68	84

Fig. 32

Generalizing, and having in mind the previous results (§ 12), we may say that

In a square of order $4t + 3$, with again an inner square of order $2t + 1$ completely filled and thus $t + 1$ borders partly filled, the p th border starting from the inner square contains $16p - 12$ empty cells, with $4p - 2$ in each of its rows ($p = 1, \dots, t + 1$).

(§ 17) Excesses and deficits in each row for the order $n = 4t + 3$

As before, the text first indicates the excesses and the deficits for each horizontal and vertical row of the successive borders for the first orders, and then the values of the increments both for successive rows within the same square and corresponding rows in successive squares. This enables us to set out a table of the differences Δ displayed by the horizontal (L) and vertical (C) rows according to both the value of t for the order $n = 4t + 3$ and the number of the border:

$n = 4t + 3$	Δ_1^L	Δ_1^C	Δ_2^L	Δ_2^C	Δ_3^L	Δ_3^C	Δ_4^L	Δ_4^C	Δ_5^L	Δ_5^C
$t = 1$	4	2	20	18						
$t = 2$	4	2	28	26	52	50				
$t = 3$	4	2	36	34	68	66	100	98		
$t = 4$	4	2	44	42	84	82	124	122	164	162

Fig. 33

From this we may infer, as before, what these differences are in general. Let us thus designate by $\Delta_p^{L,4t+3}$ and $\Delta_p^{C,4t+3}$ the excesses of the upper row and of the left-hand column in the p th border when the main square has the order $n = 4t + 3$. Then, according to the table, $\Delta_1^{L,4t+3} = 4$, $\Delta_2^{L,4t+3} = 8(t + 1) + 4$, $\Delta_3^{L,4t+3} = 16(t + 1) + 4$, $\Delta_4^{L,4t+3} = 24(t + 1) + 4$, whence, generally,

$$\Delta_p^{L,4t+3} = 8(p - 1)(t + 1) + 4 = 2(p - 1)(n + 1) + 4.$$

As for the columns of these borders, their differences are again less by 2 than those of the horizontal rows, so

$$\Delta_p^{C,4t+3} = 8(p - 1)(t + 1) + 2 = 2(p - 1)(n + 1) + 2.$$

B. Placing the even numbers

Introductory note

We now know, for each incomplete row of a border, the (even) number of empty cells it contains and its difference to the sum due. We are then to proceed as follows.

An initial placing must first eliminate the difference in each incomplete border's row ('equalization'); that is, the cells already occupied by odd numbers and those about to receive even numbers during this initial placing must then display their sum due, thus, for the order n , as many times $\frac{n^2+1}{2}$ as the number of occupied cells. But this initial placing must meet certain conditions:

- it must be uniform so that it may be applied to all squares of the same kind of order, either $n = 4t + 1$ ($t \geq 2$: order 5 has a treatment of its own) or $n = 4t + 3$;
- it must involve the least possible number of cells in order to be applicable to their lowest order, namely $n = 9$ and $n = 7$, respectively;
- it must fill the corner cells, each common to two rows;
- it must leave in each row a number of empty cells divisible by 4;

— the remaining even numbers must form groups of four, or an even number of pairs of, consecutive numbers.

If all these conditions have been met in this preliminary placing, completing the treatment is an easy matter: the rows are equalized and completing them will involve only neutral placings, that is, putting in the empty cells repeatedly and without further reflection tetrads of numbers making their sum due (see § 20).

(§ 18) Grouping the even numbers by pairs

Let us first consider the set of even numbers to be put in the square considered, as does our text. For the order $n = 2k + 1$, there are $2k^2 + 2k$ even numbers, namely $k(k + 1)$ smaller ones (less than $\frac{n^2+1}{2}$) and $k(k + 1)$ larger ones, their complements; furthermore, since $k(k + 1)$ is even, all these numbers, smaller and larger, may be arranged in pairs of consecutive even numbers. See the table below; here these pairs are numbered starting from the highest smaller numbers, the j th pair of smaller numbers having then the form $\frac{n^2-8j+3}{2}$, $\frac{n^2-8j+7}{2}$, with $j = 1, 2, \dots, \frac{1}{2}k(k + 1)$.

2	4	...	$\frac{n^2-8j+3}{2}$	$\frac{n^2-8j+7}{2}$	...	$\frac{n^2-5}{2}$	$\frac{n^2-1}{2}$
$n^2 - 1$	$n^2 - 3$	...	$\frac{n^2+8j-1}{2}$	$\frac{n^2+8j-5}{2}$	...	$\frac{n^2+7}{2}$	$\frac{n^2+3}{2}$
$\frac{1}{2}k(k + 1)$		...		j	...		1

Fig. 34

We are, as far as possible, to equalize the rows without breaking these pairs of numbers, for that will facilitate later neutral placings. Furthermore, since the difference to the sum due is the same (but with opposite signs) in two opposite rows, we may carry out the equalization for just one of two opposite rows.

Note on the equalization rules

We are now to see two main equalization rules and particular cases of them. As seen before, all the differences displayed by the rows are even, and follow one another in regular succession. We already know that we are to use, as far as possible, pairs of successive numbers, in a quantity as little as possible, to eliminate the differences left. The author's purpose is thus to show how placing pairs of even numbers may modify these differences.

(§ 19) Effect of placing a pair of small even numbers in the same row

$\frac{n^2 - 8j + 3}{2}$	$\frac{n^2 - 8j + 7}{2}$	$\Rightarrow n^2 + 1 - (8j - 4)$
$\frac{n^2 + 8j - 1}{2}$	$\frac{n^2 + 8j - 5}{2}$	$\Rightarrow n^2 + 1 + (8j - 4)$

Fig. 35

What we shall call **Rule I** tells us that placing the j th ($j \geq 1$) pair on one side and the complements opposite will produce a deficit of $8j - 4$ on the side of the j th pair and an excess of the same amount opposite (Fig. 35). That is, all remaining differences of the form 4, 12, 20, 28, ... may be eliminated using two cells provided the appropriate pair of successive even numbers is available.

Particular cases

(I', § 22) If these pairs are written in the corners (Fig. 36), that will also produce a constant difference of 2 in the two columns, which will be a deficit in that containing the lesser element.

$\frac{n^2 - 8j + 3}{2}$		$\frac{n^2 - 8j + 7}{2}$	$\Rightarrow n^2 + 1 - (8j - 4)$
$\frac{n^2 + 8j - 5}{2}$		$\frac{n^2 + 8j - 1}{2}$	$\Rightarrow n^2 + 1 + (8j - 4)$
$\Downarrow$		$\Downarrow$	
$n^2 + 1 - 2$		$n^2 + 1 + 2$	

Fig. 36

(I'', § 22) Thus, for instance, placing the largest pair in the upper corners will reduce the excess of the upper row by 4 and that of the left-hand column by 2.

(§ 21) Effect of placing two small even numbers in opposite rows

As said in the text, with an even number α written in one row then $\alpha + 2s$ in the opposite row, and next, facing them, their complements, the row of α will display a deficit of $2s$ and the opposite row an excess of $2s$ (Fig. 37).

	α	$n^2 + 1 - (\alpha + 2s)$		$\implies n^2 + 1 - 2s$
	$n^2 + 1 - \alpha$	$\alpha + 2s$		$\implies n^2 + 1 + 2s$

Fig. 37

With this Rule II, we may eliminate any even difference, thus, in fact, any difference which may occur here. But its use with $s \neq 1$ should be avoided as far as possible since we must try to keep all pairs of consecutive even numbers unbroken in order to complete the placing.

Particular cases

(II', § 30) With $s = 1$ (Fig. 38), we may eliminate an excess or a difference of 2. Since the two even numbers placed are consecutive, no pair is broken

	α	$n^2 + 1 - (\alpha + 2)$		$\implies n^2 + 1 - 2$
	$n^2 + 1 - \alpha$	$\alpha + 2$		$\implies n^2 + 1 + 2$

Fig. 38

(II'', § 29) We may also consider placing two consecutive pairs, thus four consecutive even numbers, around the border. With the complements, this will produce uniform differences of 4, which will be a deficit in the two rows containing the lesser pair (Fig. 39).

		α	$n^2 + 1 - (\alpha + 4)$		$\implies n^2 + 1 - 4$
$\alpha + 2$				$n^2 + 1 - (\alpha + 2)$	
$n^2 + 1 - (\alpha + 6)$				$\alpha + 6$	
		$n^2 + 1 - \alpha$	$\alpha + 4$		$\implies n^2 + 1 + 4$
$\Downarrow$				$\Downarrow$	
$n^2 + 1 - 4$				$n^2 + 1 + 4$	

Fig. 39

Having thus given the two main equalization rules, the text concludes (§ 22): *You must understand all this: it belongs to what you need for writing the even (numbers) in (squares of) this kind.*

Remark. As seen above, the differences to be eliminated are always even.

Now by applying once, twice or at most three times the above rules using pairs of consecutive numbers placed on one or the other side, we can eliminate differences of the form $8u$, $8u \pm 2$, $8u \pm 4$, $8u \pm 6$, that is, any difference which might occur. To do this, however, the number of empty cells available must be sufficient. If that is not the case, we shall be obliged to eliminate the difference by means of Rule II, thus using two even, non-consecutive numbers (α , $\alpha + 2s$, with $s \neq 1$); but then we shall have to apply it a second time in order to use the other two terms of the two broken pairs. See example below, § 24, in the equalization of all first (that is, smallest) borders for the order $n = 4t + 1$ (excess of the form $8u + 4$ but with four cells to be filled²¹¹). It may also happen that the difference can be eliminated with fewer cells than the quantity which is required to leave a number of empty cells divisible by 4; then we shall just use more cells than necessary (one application may only partly eliminate the difference, or even increase it). See example below, §§ 25–31, in the equalization of the other borders' horizontal rows for the order $n = 4t + 1$ (once again excess of the form $8u + 4$ but with this time eight cells available, this enabling us to use only consecutive pairs). In all other cases we shall be able to apply these rules to the least possible number of cells, and so as to leave a number of empty cells divisible by 4, thus paving the way to the neutral placings.

(§ 20) Neutral placings

Once the equalization has been effected, completing the arrangement is easy: we are to place groups of four — or groups of two pairs of — even consecutive numbers, with the extremes on one side and the means opposite. The four numbers will thus have the form α , $\alpha + 2$, $\alpha + 4$, $\alpha + 6$ if they are consecutive (Fig. 40) or α , $\alpha + 2$, β , $\beta + 2$ if they form two pairs (§ 31).

	α	$n^2 + 1 - (\alpha + 2)$	$n^2 + 1 - (\alpha + 4)$	$\alpha + 6$		$\Rightarrow 2(n^2 + 1)$
	$n^2 + 1 - \alpha$	$\alpha + 2$	$\alpha + 4$	$n^2 + 1 - (\alpha + 6)$		$\Rightarrow 2(n^2 + 1)$

Fig. 40

²¹¹ With *two* cells the choice of the pair $j = u + 1$ would be appropriate: see §§ 32–33.

C. Completing the squares for the three order types

Introductory note

As seen before (§12), there are for the filling of odd-order squares displaying the separation by parity three categories. That of order 5, which is a particular case, that of the further squares with $n = 4t + 1$, and that of the squares with $n = 4t + 3$. The first will be treated in the coming paragraph, the second category in §§24–31, and the third in §§32–35.

(§23) Square of order 5

As observed in §12, the general construction for orders of the type $n = 4t + 1$ (represented in our Fig. 49 further below, p. 194) cannot apply to the smallest order, with $t = 1$. Indeed, since in this case $\frac{n^2-5}{2}$ and $\frac{n^2-1}{2}$ happen to be numerically equal to, respectively, $2n$ and $2n + 2$, the even numbers supposed to occupy the upper corners would also be found within the border. Our text gives therefore a particular construction for this square, with the result as in our Fig. 41 (orientation changed and inner square inverted).

2	16	25	18	4
6	7	21	11	20
23	17	13	9	3
12	15	5	19	14
22	10	1	8	24

Fig. 41

Whereas our text merely indicates where to put each number, without any justification, Būzjānī observes that if we place 2 in a corner, the next corner (horizontally or vertically, if we keep 1 below) must be occupied by the (smaller) number 4, 6, 8 or 12, which leads to ‘numerous figures’.²¹² As a matter of fact, this gives eleven figures. But if we do not keep 2 in a corner, there are altogether twenty-one possibilities with the first twenty-five numbers.²¹³

²¹² *Magic squares in the tenth century*, B.22, ii; or edition of the full text, pp. 177 & 233–234.

²¹³ Listed (including the previous eleven) in *Magic squares*, p. 209 (*Les carrés magiques*, p. 178; *Маг. квадраты*, p. 190). Furthermore, considering permutations of the even numbers within the border rows together with the eight aspects resulting from inverting and rotating the inner 3×3 square will lead to $21 \cdot 4 \cdot 8 = 672$ configurations for this 5×5 square.

Squares of orders $n = 4t + 1$, $t > 1$

Since the bottom rows are to receive the complements, we shall consider only the top ones, thus those in excess; likewise, we shall consider the left-hand rows, also in excess. As we have seen (§ 13), the rows of the p th border ($p = 1, \dots, t$, counting from the inner square already filled) comprise $4p$ empty cells. Eliminating the differences must thus be effected with four cells for the rows of all first borders, whereas we may use eight cells for the rows of the other borders, this being applicable to the outer border of the lowest order $n = 9$ as well.

(§ 24) Filling all first borders

As just said, only four cells are available, and the excess in the upper rows is of the form $\Delta_1^{L, 4t+1} = 8t + 4$ (§ 14). We are thus to eliminate this excess by means of two pairs of numbers. This will be done as follows (Fig. 42).

$\frac{n^2-5}{2}$	2		$n^2+1-\Delta_1^C$	$\frac{n^2-1}{2}$
4				n^2+1-4
$n^2+1-\Delta_1^L$				Δ_1^L
$\frac{n^2+3}{2}$	n^2+1-2		Δ_1^C	$\frac{n^2+7}{2}$

Fig. 42

Let us take for the first pair the largest dyad ($j = 1$), namely $\frac{n^2-5}{2}$ and $\frac{n^2-1}{2}$, and put it in the corners, say in the left-hand and right-hand corners, respectively. Since their sum is $n^2 - 3$, they display a deficit of 4 relative to their sum due $n^2 + 1$ (according to Rule I'), and this leaves us with eliminating $8t$ by means of two numbers. The only way is, in conformity with Rule II, to use two numbers from different dyads, say the smallest number 2 and thus the larger number $n^2 + 1 - (8t + 2)$, which happens to be $n^2 + 1 - \Delta_1^C$; their sum eliminates the excess completely and fills the last two available cells. This is just what the text says, except that it is $8t + 2$ (the $4t$ th even number after 2, as the extant text expresses it) which is said to be put in the opposite row. See, for the values, the table in Fig. 43 and, for the squares, Fig. 46–48.

Consider now the columns. The left-hand corner cells are now occupied by the smaller of the numbers placed first and the complement of the other, thus by $\frac{n^2-5}{2}$, $\frac{n^2+3}{2}$, the sum of which is $n^2 - 1$; the initial

excess on the left-hand side, $\Delta_1^C = 8t + 2$, is thus reduced to $8t$ (again in accordance with Rule I'). Since there are only two empty cells available, we must again eliminate this excess, according to Rule II, by two numbers from different dyads. Putting 4 in the (for us) left-hand column and, in the right-hand one, $8t + 4$ (thus Δ_1^L), not only shall we have eliminated the remaining excess $8t$, since the left-hand column now contains 4 and $n^2 + 1 - (8t + 4)$, but we shall also have placed the remaining terms of the two pairs previously broken, thereby avoiding the problem of using subsequently two single numbers. See Fig. 43 and Fig. 46–48.

	$\frac{n^2-5}{2}$	$\frac{n^2-1}{2}$	2	$8t+2$	4	$8t+4$
$n = 9 \quad (t = 2)$	38	40	2	18	4	20
$n = 13 \quad (t = 3)$	82	84	2	26	4	28
$n = 17 \quad (t = 4)$	142	144	2	34	4	36

Fig. 43

(§§ 25–28) Obtaining uniform excesses and deficits in all other borders

According to what precedes (§§ 13–14), the excesses in the upper rows are $\Delta_p^{L, 4t+1} = 8pt + 4$, and there are $4p$ empty cells. A direct elimination by one dyad, thus with two cells (in this case corner cells), would be possible, namely taking $j = pt + 1$; but it would be unsuitable since the number of cells left empty would not be divisible by 4. On the other hand, elimination by two dyads, thus using four cells, is not possible. Therefore we shall eliminate these excesses by means of eight cells, this being also applicable to the smallest order $n = 9$. The same will be done with the excesses in the left-hand columns, originally $\Delta_p^{C, 4t+1} = 8pt + 2$. All that will be reached in five steps (*i-v* below). In the first (§ 25*i*, § 26*i*, § 27*i*), we shall fill the two corner cells of each p th column in excess (for us the left-hand ones) by means of a consecutive even pair $\alpha_p, \alpha_p + 2$. In the second (§ 25*ii*, § 26*ii*, § 27*ii*), two consecutive numbers $\beta_p, \beta_p + 2$ put within these columns will reduce their excesses to a uniform quantity. In the third step (§ 25*iii*, § 26*iii*, § 27*iii*), this same situation will be attained for the horizontal rows, placing pairs $\gamma_p, \gamma_p + 2$ in two of the six cells still available. All this relies on the use of Rule I and Rule I'. The last two equalization steps, applied together to horizontal rows and columns, will remove, by using the remaining four cells, the constant differences left.

(i) Filling the corners of the column in excess

We first look for two even numbers, α_p and $\alpha_p + 2$, adding up to $(n^2 + 1) - (8p - 4)$. Therefore, these numbers will be

$$\alpha_p = \frac{n^2 - 8p + 3}{2}, \quad \alpha_p + 2 = \frac{n^2 - 8p + 7}{2},$$

the lesser of them being put in the upper corner (Fig. 44). For these placings, the text explains that we are to put, at either end of each of these columns, a pair of consecutive small numbers the sum of which is, respectively, less than their sum due by 12, 20, 28, that is, by our $8p - 4$ (this subtractive quantity being thus the same for same borders). See the resulting values for the first orders in Fig. 45, and, for their placing, Fig. 46–48. Note that the above formulas are valid for the first border as well ($p = 1$), but with the pair being placed in the upper row.

α_p		γ_p	$\gamma_p + 2$	$n^2 + 1 - (\alpha_p + 2)$
β_p				$n^2 + 1 - \beta_p$
$\beta_p + 2$				$n^2 + 1 - (\beta_p + 2)$
$\alpha_p + 2$		$n^2 + 1 - \gamma_p$	$n^2 + 1 - (\gamma_p + 2)$	$n^2 + 1 - \alpha_p$

Fig. 44

	α_p	$\alpha_p + 2$	β_p	$\beta_p + 2$	γ_p	$\gamma_p + 2$
$n = 9, \quad p = 2$	34	36	30	32	26	28
$n = 13, \quad p = 2$	78	80	66	68	62	64
$\quad \quad \quad p = 3$	74	76	58	60	50	52
$n = 17, \quad p = 2$	138	140	118	120	114	116
$\quad \quad \quad p = 3$	134	136	106	108	98	100
$\quad \quad \quad p = 4$	130	132	94	96	82	84

Fig. 45

(ii) Reducing the excesses in the columns to a uniform value

With these two numbers, adding up to $(n^2 + 1) - (8p - 4)$, the initial excess of the p th left-hand column $\Delta_p^{C, 4t+1} = 8pt + 2$ has become $8p(t - 1) + 6$. Now we infer from the first main equalization rule how an excess of such a form may be treated: putting $8p(t - 1) + 6 = 8j - 4$, we have $8j = 8p(t - 1) + 10$; taking the closest integral value, $j = p(t - 1) + 1$, will leave a uniform remainder of 2. We shall therefore put in the p th left-hand column ($p \geq 2$) the pair

$$\beta_p = \frac{n^2 - 8[p(t - 1) + 1] + 3}{2} = \frac{n^2 - 2pn + 10p - 5}{2}$$

$$\beta_p + 2 = \frac{n^2 - 8[p(t - 1) + 1] + 7}{2} = \frac{n^2 - 2pn + 10p - 1}{2}.$$

These values for the first orders are seen in the above table, and, for the corresponding squares, in Fig. 46–48. We are then left with an excess of 2 in each left-hand column and have four cells to complete the equalization.

For the above placings, the text explains that we are to put in the (for us left-hand) columns pairs of consecutive small numbers adding up to their sum due less an amount equal to the (initial) excess of the column minus, respectively, 14, 22, 30, thus minus the quantity $8p-2$; that is, the sum of these two numbers β_p, β_p+2 must be $(n^2+1) - [8pt+2 - (8p-2)] = n^2 - 8[p(t-1) + 1] + 5$, whence the above pair of values.

(iii) Reducing the excesses in the upper rows to a uniform value

Consider now the upper rows. Their initial excesses have changed since one number has been placed in each p th left-hand corner while the right-hand one is occupied by a complement. Since these two numbers, namely

$$\frac{n^2 - 8p + 3}{2}, \quad \frac{n^2 + 8p - 5}{2},$$

add up to $n^2 - 1 = (n^2 + 1) - 2$, the initial excesses in the upper rows have been reduced from $8pt + 4$ to $8pt + 2$, and there are $4p - 2$ empty cells. Putting, in the same manner as before, $8pt + 2 = 8j - 4$, we are led to take $j = pt$ and thus place within the p th upper row, the pt th pair of smaller numbers

$$\gamma_p = \frac{n^2 - 8pt + 3}{2} = \frac{n^2 - 2pn + 2p + 3}{2}$$

$$\gamma_p + 2 = \frac{n^2 - 8pt + 7}{2} = \frac{n^2 - 2pn + 2p + 7}{2}.$$

These values for the first orders are seen in the above table, and in Fig. 46–48 for the squares. This then leaves an excess of 6 in each upper row and four cells to complete the equalization. The text expresses the choice of these numbers in the same way as before: we are to put in the p th upper row two consecutive even smaller numbers, say γ_p and $\gamma_p + 2$, the sum of which is less than their sum due by the (initial) excess of the upper row minus 8 (thus minus a quantity independent of border and order); that is, their sum must be $2\gamma_p + 2 = (n^2 + 1) - [(8pt + 4) - 8]$, which gives the above two values.

With this and what we have found for the first border, we can now set out the table in Fig. 49 (the asterisk marks cells to receive complements). Although it gives the required quantities for only five borders, it clearly shows how it may be extended: the terms belonging to the same oblique line can be found using arithmetical progressions —the only exceptions

being in the first border. That is therefore sufficient for constructing any square of order $n = 4t + 1$ with separation by parity; indeed, what remains to complete its rows is straightforward.

34	26	28		77					46
30	38	2	3	69	71	64	40	52	
32	4	23	63	61	31	27	78	50	
	73	53	35	49	39	29	9		
75	67	57	45	41	37	25	15	7	
	1	17	43	33	47	65	81		
	62	55	19	21	51	59	20		
	42	80	79	13	11	18	44		
36	56	54		5					48

Fig. 46

74	50	52				161					94
58	78	62	64		7	145	155			90	112
60	66	82	2	3	23	137	139	151	144	84	104
	68	4	47	131	127	125	55	59	51	166	102
		153	113	67	107	105	75	71	57	17	
	157	141	117	97	79	93	83	73	53	29	13
159	143	135	121	101	89	85	81	69	49	35	27
	5	21	41	61	87	77	91	109	129	149	165
		1	37	99	63	65	95	103	133	169	
		142	119	39	43	45	115	111	123	28	
		86	168	167	147	33	31	19	26	88	
	80	108	106		163	25	15			92	
76	120	118				9					96

Fig. 47

130	82	84						277							158
94	134	98	100				11	253	271					154	196
96	106	138	114	116		7	35	237	247	267				150	184
	108	118	142	2	3	31	51	229	231	243	263	256	144	172	182
		120	4	79	223	219	215	213	87	91	95	83	286	170	
			265	197	107	191	187	185	115	119	111	93	25		
		269	245	201	173	127	167	165	135	131	117	89	45	21	
	273	249	233	205	177	157	139	153	143	133	113	85	57	41	17
275	251	235	227	209	181	161	149	145	141	129	109	81	63	55	39
	9	33	49	73	101	121	147	137	151	169	189	217	241	257	281
		5	29	69	97	159	123	125	155	163	193	221	261	285	
			1	65	179	99	103	105	175	171	183	225	289		
			254	207	67	71	75	77	203	199	195	211	36		
			146	288	287	259	239	61	59	47	27	34	148		
		140	176	174		283	255	53	43	23				152	
	136	192	190			279	37	19						156	
132	208	206						13							160

Fig. 48

(§§ 29–31) Completing the equalization

As stated in the text (§ 28) there now remains — except for the rows of the first border, which are completely filled (and equalized) — a number of cells divisible by 4 in each row and uniform excesses of 6 in the upper ones and of 2 in the left-hand columns (Fig. 50). In order to have an equalization valid from the smallest order (here $n = 9$), we are to fill four cells in each row. We further know that the numbers still available all form groups of four, or pairs of dyads of, consecutive even numbers, for we have already placed the two pairs which had been broken.

Eliminating the remaining excesses will be carried out in the next two steps.

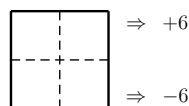


Fig. 50

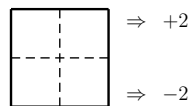


Fig. 51

(iv) First (§ 29), using Rule II'' (p. 186), we put four consecutive (even) numbers around each border, beginning on the top and turning (for us) towards the left.

	δ	$n^2 + 1 - (\delta + 4)$	
$\delta + 2$			$n^2 + 1 - (\delta + 2)$
$n^2 + 1 - (\delta + 6)$			$\delta + 6$
	$n^2 + 1 - \delta$	$\delta + 4$	

Fig. 52

As we know from this rule, and from Fig. 52, with the complements in place, that will change the excesses and deficits, formerly ± 6 in the horizontal rows and ± 2 in the columns (Fig. 50), to ± 2 and ∓ 2 (Fig. 51). In Fig. 54 ($n = 9$, with $p = 2$) we have thus placed, starting in the upper row, the groups of four numbers 6, 8, 10, 12; in Fig. 55 ($n = 13$) the groups 6, 8, 10, 12 ($p = 3$) and 30, 32, 34, 36 ($p = 2$); in Fig. 56 ($n = 17$) the groups 6, 8, 10, 12 ($p = 4$), then 22, 24, 26, 28 ($p = 3$) and 42, 44, 46, 48 ($p = 2$). Our text also constructs the three examples with

$n = 9, 13, 17.$ ²¹⁴

(v) Second (§30), using Rule II', thus the simplest case ($s = 1$) of the second equalization rule, take any two pairs of available consecutive numbers and place each in a pair of opposite rows (Fig. 53), with the lesser term on the side of the excess.

	ϵ	$n^2 + 1 - (\epsilon + 2)$	
	$n^2 + 1 - \epsilon$	$\epsilon + 2$	

Fig. 53

With that done for all borders, and the complements written in, all remaining differences will be eliminated. In Fig. 54 ($n = 9$), we have thus placed 14, 16 and 22, 24 ($p = 2$); in Fig. 55 ($n = 13$) 14, 16 and 18, 20 ($p = 3$), then 22, 24 and 38, 40 ($p = 2$); in Fig. 56 ($n = 17$) 14, 16 and 18, 20 ($p = 4$), then 30, 32 and 38, 40 ($p = 3$), finally 50, 52 and 54, 56 ($p = 2$).

34	26	28	6	77	72	14	66	46
30	38	2	3	69	71	64	40	52
32	4	23	63	61	31	27	78	50
8	73	53	35	49	39	29	9	74
75	67	57	45	41	37	25	15	7
70	1	17	43	33	47	65	81	12
60	62	55	19	21	51	59	20	22
24	42	80	79	13	11	18	44	58
36	56	54	76	5	10	68	16	48

Fig. 54

74	50	52	6	160	14	161	154	42	126	124	48	94
58	78	62	64	30	7	145	155	136	22	146	90	112
60	66	82	2	3	23	137	139	151	144	84	104	110
8	68	4	47	131	127	125	55	59	51	166	102	162
158	32	153	113	67	107	105	75	71	57	17	138	12
152	157	141	117	97	79	93	83	73	53	29	13	18
159	143	135	121	101	89	85	81	69	49	35	27	11
20	5	21	41	61	87	77	91	109	129	149	165	150
54	134	1	37	99	63	65	95	103	133	169	36	116
114	132	142	119	39	43	45	115	111	123	28	38	56
100	40	86	168	167	147	33	31	19	26	88	130	70
72	80	108	106	140	163	25	15	34	148	24	92	98
76	120	118	164	10	156	9	16	128	44	46	122	96

Fig. 55

As asserted (§31) we have now completed the first two borders around the central square and are left in the subsequent borders ($p \geq 3$) with numbers of empty cells all divisible by 4. Since the rows of these borders

²¹⁴ The difference with the squares actually presented by our text is due to the fact that the author fills the cells from the first inside border and by taking the available small numbers in decreasing order. In both cases the numbers chosen just fill up progressively the gaps left by the former placings in the continuous number sequence.

display neither excesses nor deficits, we may fill them repeatedly with neutral placings, thus either four consecutive even numbers or two separate pairs of even numbers, of which we write the extreme terms on one side and the middle terms on the other, the complements providing then the sum due for four cells (above, § 20). In Fig. 55 ($n = 13$), 42, ..., 48 and the pairs 54, 56 & 70, 72 complete the outer border (50, 52 and 58, ..., 68 have been used for the preliminary placing); in Fig. 56 ($n = 17$), the two upper rows and the ones opposite have been completed with the tetrads 58, ..., 64, 66, ..., 72 and 74, ..., 80, respectively, the columns with 86, ..., 92 and the pairs 102, 104 & 110, 112, and 122, ..., 128, respectively.

130	82	84	6	280	14	274	58	277	230	228	64	66	222	220	72	158
94	134	98	100	22	264	30	11	253	271	258	74	214	212	80	154	196
96	106	138	114	116	42	7	35	237	247	267	244	50	238	150	184	194
8	108	118	142	2	3	31	51	229	231	243	263	256	144	172	182	282
278	24	120	4	79	223	219	215	213	87	91	95	83	286	170	266	12
272	262	44	265	197	107	191	187	185	115	119	111	93	25	246	28	18
20	252	269	245	201	173	127	167	165	135	131	117	89	45	21	38	270
86	273	249	233	205	177	157	139	153	143	133	113	85	57	41	17	204
275	251	235	227	209	181	161	149	145	141	129	109	81	63	55	39	15
202	9	33	49	73	101	121	147	137	151	169	189	217	241	257	281	88
200	40	5	29	69	97	159	123	125	155	163	193	221	261	285	250	90
92	122	242	1	65	179	99	103	105	175	171	183	225	289	48	168	198
102	166	236	254	207	67	71	75	77	203	199	195	211	36	54	124	188
186	164	56	146	288	287	259	239	61	59	47	27	34	148	234	126	104
180	128	140	176	174	248	283	255	53	43	23	46	240	52	152	162	110
112	136	192	190	268	26	260	279	37	19	32	216	76	78	210	156	178
132	208	206	284	10	276	16	232	13	60	62	226	224	68	70	218	160

Fig. 56

Squares of orders $n = 4t + 3$

We shall again consider only the rows in excess, thus the upper rows and the (for us) left-hand columns.

(§ 32) Completing all first borders

As already seen (§§ 16–17, pp. 182–183), the number of empty cells in each row of the p th border is $4p - 2$ and the excesses are $\Delta_p^{L, 4t+3} = 8(p-1)(t+1) + 4$ and $\Delta_p^{C, 4t+3} = 8(p-1)(t+1) + 2$. Thus, for all first borders the equalization *must* be effected with two cells.

Since the excesses are $\Delta_1^{L, 4t+3} = 4$ for the upper rows and $\Delta_1^{C, 4t+3} = 2$ for the columns, we shall apply the particular case I'' (§ 22, p. 185). That is, we put in the upper corners of the first row the largest pair of smaller numbers, thus

$$\frac{n^2 - 5}{2}, \quad \frac{n^2 - 1}{2},$$

with the larger one in our right-hand corner, where the column displays a deficit. Then we shall have, as stated in the text, equalized all first borders. Note that the same pair had been used for the first borders in the case $n = 4t + 1$ —though without then completing the equalization. In Fig. 59 ($n = 7$) we have thus placed in the upper corners of the first border 22, 24; in Fig. 60 ($n = 11$), 58, 60; in Fig. 61 ($n = 15$), 110, 112.

(§ 33) Equalizing the horizontal rows of all other borders

A single pair will turn out to be again sufficient for all upper rows of larger borders ($p \geq 2$), while for the columns, of which two corner cells are now occupied, at least four cells will be available.

α_p		$\alpha_p + 2$
γ_p		$n^2 + 1 - \gamma_p$
$\gamma_p - 2$		$n^2 + 1 - (\gamma_p - 2)$
β_p		$n^2 + 1 - \beta_p$
$\beta_p + 2$		$n^2 + 1 - (\beta_p + 2)$
$n^2 + 1 - (\alpha_p + 2)$		$n^2 + 1 - \alpha_p$

Fig. 57

Since the excess in the upper rows is $\Delta_p^{L, 4t+3} = 8(p-1)(t+1) + 4$, we shall choose, as stated in the text, a pair of (consecutive) small numbers less than their sum due, $n^2 + 1$, by that amount:

$$\alpha_p + \alpha_p + 2 = n^2 + 1 - [8(p-1)(t+1) + 4].$$

Thus, applying Rule I' (p. 185; cf. p. 187*n*), we shall take $j = (p - 1)(t + 1) + 1$ and write in the end cells of the p th upper row ($p \geq 2$) the corresponding pair, that is,

$$\alpha_p = \frac{n^2 - 8[(p - 1)(t + 1) + 1] + 3}{2} = \frac{n^2 - 2(p - 1)n - 2p - 3}{2},$$

$$\alpha_p + 2 = \frac{n^2 - 8[(p - 1)(t + 1) + 1] + 7}{2} = \frac{n^2 - 2(p - 1)n - 2p + 1}{2},$$

with the lesser term in the column with an excess. (Note that the pair for $p = 1$ relies on the same formula.) See Fig. 58, and Fig. 59–61 for the squares of orders $n = 7, 11, 15$. As then stated in the text (§ 33, *in fine*), we are left with completing the horizontal rows with neutral placings.

	α_p	$\alpha_p + 2$	$\gamma_p - 2$	γ_p	β_p	$\beta_p + 2$
$n = 7, \quad p = 2$	14	16	30	32	10	12
$n = 11, \quad p = 2$	46	48	66	68	42	44
$\quad \quad \quad p = 3$	34	36	70	72	26	28
$n = 15, \quad p = 2$	94	96	118	120	90	92
$\quad \quad \quad p = 3$	78	80	122	124	70	72
$\quad \quad \quad p = 4$	62	64	126	128	50	52
$n = 19, \quad p = 2$	158	160	186	188	154	156
$\quad \quad \quad p = 3$	138	140	190	192	130	132
$\quad \quad \quad p = 4$	118	120	194	196	106	108
$\quad \quad \quad p = 5$	98	100	198	200	82	84

Fig. 58

(§ 34) Equalizing the vertical rows of all other borders

With the above placing, the situation in the left-hand columns has changed, for their corner cells are now occupied by the first number of the pair above and the complement of the other, the sum of which is $(n^2 + 1) - 2$. There remains then on the p th left-hand side an excess of $8(p - 1)(t + 1)$ while the number of still empty cells is now 4 ($p = 2$) or a multiple of 4 ($p \geq 3$). We shall then eliminate the differences with four numbers.

But here the text preserved is imprecise, for it gives a single condition for determining the two pairs of consecutive numbers to be put in the (for us) left-hand columns, two larger and two smaller, leaving thus one number (or pair) optional. As a matter of fact, since the corners opposite to those filled in the first border are occupied by the two larger complements $\frac{n^2+7}{2}$ and $\frac{n^2+3}{2}$, we shall just take, as the pairs of larger numbers, the subsequent ones, namely $\frac{n^2+15}{2}$, $\frac{n^2+11}{2}$; $\frac{n^2+23}{2}$, $\frac{n^2+19}{2}$; That is,

generally, the pairs of larger numbers to be put in the left-hand columns of the p th border will be

$$\gamma_p = \frac{n^2 + 8p - 1}{2}, \quad \gamma_p - 2 = \frac{n^2 + 8p - 5}{2},$$

the sum of which is $n^2 + 8p - 3 = (n^2 + 1) + 8p - 4$.

14				45			16
30	22	3	37	39	24		20
32	41	19	33	23	9		18
43	35	29	25	21	15	7	
10	1	27	17	31	49		40
12	26	47	13	11	28		38
34				5			36

Fig. 59

34					113					36
70	46			7	97	107			48	52
72	66	58	3	23	89	91	103	60	56	50
26	68	105	43	83	81	51	47	17	54	96
28	109	93	73	55	69	59	49	29	13	94
111	95	87	77	65	61	57	45	35	27	11
	5	21	37	63	53	67	85	101	117	
	42	1	75	39	41	71	79	121	80	
	44	62	119	99	33	31	19	64	78	
	74			115	25	15			76	
86					9					88

Fig. 60

62								213							64
126	78					11	189	207						80	100
128	122	94			7	35	173	183	203				96	104	98
50	124	118	110	3	31	51	165	167	179	199	112	108	102	176	
52	70	120	201	75	159	155	153	83	87	79	25	106	156	174	
	72	205	181	141	95	135	133	103	99	85	45	21	154		
	209	185	169	145	125	107	121	111	101	81	57	41	17		
211	187	171	163	149	129	117	113	109	97	77	63	55	39	15	
	9	33	49	69	89	115	105	119	137	157	177	193	217		
		5	29	65	127	91	93	123	131	161	197	221			
		90	1	147	67	71	73	143	139	151	225	136			
		92	114	223	195	175	61	59	47	27	116	134			
		130			219	191	53	43	23			132			
	146					215	37	19					148		
162							13								164

Fig. 61

The two larger numbers being thus set, the pair of smaller numbers to be put with them is easy to find. Since the excess in the p th left-hand column has now increased to $8(p-1)(t+1)+8p-4 = 8[(p-1)(t+1)+p]-4$, we shall just put, according to Rule I (p. 185), $j = (p-1)(t+1) + p$, and thus take as the pair of smaller numbers

$$\beta_p = \frac{n^2 - 8[(p-1)(t+1) + p] + 3}{2} = \frac{n^2 - 2(p-1)n - 10p + 5}{2},$$

$$\beta_p + 2 = \frac{n^2 - 8[(p-1)(t+1) + p] + 7}{2} = \frac{n^2 - 2(p-1)n - 10p + 9}{2}.$$

Once these four numbers — two larger and two smaller — have been placed in the (for us) left-hand columns, all their excesses are eliminated, and so also in the right-hand columns after placing of the complements. See the values in the table above, or Fig. 59 ($n = 7$), Fig. 60 ($n = 11$), Fig. 61 ($n = 15$).

Knowing all these relations we may now set out Fig. 62 (p. 203), which shows the placings required for the first five borders. As before (Fig. 49), that enables us to fill the squares for any order of this type since determining the terms of the preliminary placing becomes evident.

Since now each row displays the sum due for the number of cells filled, it can be completed with neutral placings. In Fig. 63 ($n = 7$), a single group of four numbers $2, \dots, 8$ completes the square; in Fig. 64 ($n = 11$), $2, \dots, 16$, and, for the columns, the pairs $30, 32$ & $38, 40$ ($p = 3$), next $18, \dots, 24$ ($p = 2$); in Fig. 65 ($n = 15$), $2, \dots, 24$ and, for the columns, $54, \dots, 60$, then the pairs $66, 68$ & $74, 76$ ($p = 4$), next $26, \dots, 40$ and $82, \dots, 88$ ($p = 3$), finally, $42, \dots, 48$ ($p = 2$).

14	2	46	45	44	8	16
30	22	3	37	39	24	20
32	41	19	33	23	9	18
43	35	29	25	21	15	7
10	1	27	17	31	49	40
12	26	47	13	11	28	38
34	48	4	5	6	42	36

Fig. 63

34	2	118	116	8	113	10	110	108	16	36
70	46	18	102	7	97	107	100	24	48	52
72	66	58	3	23	89	91	103	60	56	50
26	68	105	43	83	81	51	47	17	54	96
28	109	93	73	55	69	59	49	29	13	94
111	95	87	77	65	61	57	45	35	27	11
30	5	21	37	63	53	67	85	101	117	92
90	42	1	75	39	41	71	79	121	80	32
84	44	62	119	99	33	31	19	64	78	38
40	74	104	20	115	25	15	22	98	76	82
86	120	4	6	114	9	112	12	14	106	88

Fig. 64

62	2	222	220	8	10	214	213	212	16	18	206	204	24	64
66	78	26	198	196	32	11	189	207	34	190	188	40	80	160
158	54	94	42	182	7	35	173	183	203	180	48	96	172	68
152	170	120	110	3	31	51	165	167	179	199	112	106	56	74
76	168	118	201	75	159	155	153	83	87	79	25	108	58	150
82	60	205	181	141	95	135	133	103	99	85	45	21	166	144
142	209	185	169	145	125	111	121	107	107	81	57	41	17	84
211	187	171	163	149	129	109	113	117	97	77	63	55	39	15
140	9	33	49	69	89	119	105	115	137	157	177	193	217	86
88	124	5	29	65	127	91	93	123	131	161	197	221	102	138
128	122	90	1	147	67	71	73	143	139	151	225	136	104	98
126	70	92	114	223	195	175	61	59	47	27	116	134	156	100
50	72	130	184	44	219	191	53	43	23	46	178	132	154	176
52	146	200	28	30	194	215	37	19	192	36	38	186	148	174
162	224	4	6	218	216	12	13	14	210	208	20	22	202	164

Fig. 65

(§ 35) Conclusion

The theory ends in our text with an example, that of constructing the column of order 7.

The treatise then represents the squares from $n = 5$ to $n = 19$ (above, pp. 76–87).²¹⁵ They may differ from those we have just constructed: as already said (p. 196*n*), the author fills the incomplete borders from the first, innermost one and by taking the available small numbers in decreasing order.

In our transcription of these examples, we have been faithful to the Arabic, except for changing the orientation (reading from left to right, as was probably the case in the original text). Note also that in all these text's examples the inner 3×3 square appears inverted around the ascending diagonal, with its smallest element on the vertical side; this must have been made to be in keeping with the situation in the larger odd borders.

²¹⁵ Of these, Anṭākī reports the squares for $n = 5$ and $n = 9$, then $n = 7$ and $n = 11$ (those for $n = 5, 7$ with Indo-Arabic numerals, see p. 27*n*).

$\frac{n^2-8n-13}{2}$																			$\frac{n^2-8n-9}{2}$
$\frac{n^2+35}{2}$	$\frac{n^2-6n-11}{2}$																	$\frac{n^2-6n-7}{2}$	*
$\frac{n^2+39}{2}$	$\frac{n^2+27}{2}$	$\frac{n^2-4n-9}{2}$																$\frac{n^2-4n-5}{2}$	*
$\frac{n^2-8n-45}{2}$	$\frac{n^2+31}{2}$	$\frac{n^2+19}{2}$	$\frac{n^2-2n-7}{2}$															$\frac{n^2-2n-3}{2}$	*
$\frac{n^2-8n-41}{2}$	$\frac{n^2-6n-35}{2}$	$\frac{n^2+23}{2}$	$\frac{n^2+11}{2}$	$\frac{n^2-5}{2}$														$\frac{n^2-1}{2}$	*
	$\frac{n^2-6n-31}{2}$	$\frac{n^2-4n-25}{2}$	$\frac{n^2+15}{2}$															*	
		$\frac{n^2-4n-21}{2}$																*	
			$\frac{n^2-2n-15}{2}$															*	*
			$\frac{n^2-2n-11}{2}$	*														*	*
			*															*	
		*																	
*																			*

Fig. 62

Part III: Construction of even-order bordered squares

(§ 36) The three categories of even-order squares

There are, apart from the particular case of order 4, three categories of even squares. Those of evenly-odd orders ($n = 4k + 2$), those of (so-called) evenly-even orders ($n = 8t$), and those of evenly-evenly-odd orders ($n = 8t + 4$), with k, t natural. Their constructions will be taught now.

But, as we shall see, the constructions of the last two, which group the even orders divisible by 4 ($n = 4k$, usually called properly ‘evenly-even orders’), are (slightly) different only for the smallest square $n = 8$. On the other hand, neither construction applies to the 4×4 square: since there is no magic square of order 2 with different numbers, the border of order 4 cannot be filled separately, that is, the 4×4 square must be filled as a whole.

(§ 37) Construction of the square of order 4

As already said, just as the smallest inner square in the bordered squares of odd orders is that of order 3, the smallest inner square for even orders is that of order 4. In our treatise, its construction is taught, but not justified. The resulting square (here from left to right) is as in Fig. 66.

4	15	10	5
14	1	8	11
7	12	13	2
9	6	3	16

Fig. 66

Note on this construction

After placing 1 in the second horizontal row, we descend first by the knight’s move, then move away from the side by a queen’s move, then resume with the knight’s move to the next (here top) row. For the other sequence of four numbers, beginning with 8, we descend symmetrically with decreasing numbers.

It then appears that each horizontal row contains the same sum, 9, and two alternate columns also the same sum, 7 or 11. It also appears that each bishop’s cell of a cell filled has remained empty. We shall fill them in such a way that the sum of two conjugate cells — that is, joined by a bishop’s move — make $n^2 + 1 = 17$. Since all horizontal rows

contain, after the preliminary filling, the same sum, the complements of any horizontal row will produce the magic sum in any other, in particular in its conjugate. As for the columns, the sums in them are not always the same but they are in two conjugate columns, so when filled they will display the magic sum. The remaining case, that of the diagonals, is clear since they contain pairs of complements.

As noted earlier (p. 3), the possibilities for a square of order 4 are numerous. With this construction, the resulting square will be ‘pandiagonal’, which is to say that not only the two main diagonals will make the required sum, but also any broken diagonal, that is, two complementary parts on either side of a main diagonal. Thus, in Fig. 66, the sets 7, 1, 10, 16 and 10, 11, 7, 6 will each add up to $M_4 = 34$. Now the property of a pandiagonal magic square is that it will remain magic if we move any lateral row to the other side: the main diagonals of the new square will be magic since they were already magic as broken diagonals. Thus, by repeating such vertical and/or horizontal moves we are able to place any chosen element in any given cell of the square; for example, the cell containing 1 can be made to occupy any place in the square. Now, as we shall see, the configuration adopted by our treatise for squares of even orders is to have the order number in an upper corner cell. This will also be the case for the square of order 4 by putting 1 (and the smaller median) in the second horizontal row, whence this particular choice.

A. Bordered squares of evenly-odd orders ($n = 4k + 2$)

(§§ 38–39) Method of construction

The method of construction described in our treatise is illustrated by the first two squares, for $n = 6$ and $n = 10$ (Fig. 67–68). To place the smaller numbers in this way, our text begins by filling the inner 4×4 square, starting, as usual, with the two medians, and then continuing for the 6×6 border with the (smaller) numbers taken in decreasing order. The differentiation by parity is pointed out: the two columns are filled with sequences of small (mainly) even numbers, and the two horizontal rows with sequences of small (mainly) odd numbers.

If, instead of taking the numbers in decreasing order, as does the text, we start in the outer border with the smallest number, say 1 here, the general method can be described as follows. We write 1 in the bottom row, next to the lower left-hand corner, then 2 above, 3 below, and the next ones cyclically around the border until we reach $4k = n - 2$ (thus, here, 4 and 8, respectively), which will be in a cell of the right-hand row

if we have adopted the anti-clockwise movement. We then write the next two numbers, $n - 1$ and n (here 5, 6 and 9, 10) in the upper left and right corners. So far, we have written $4k + 2$ small numbers. Next, we put the two following numbers on the left, then the subsequent ones around the border, in the same rotating movement as before, until we reach on the right side (for us), after thus placing $4k$ more numbers, $8k + 2 = 2n - 2$ (10 and 18 here). Half the border cells are thus filled, and putting their complements opposite will fill the border completely.

5	36	2	34	28	6
27	14	25	20	15	10
8	24	11	18	21	29
7	17	22	23	12	30
33	19	16	13	26	4
31	1	35	3	9	32

Fig. 67

9	100	2	98	5	94	88	15	84	10
83	25	32	31	71	72	73	74	26	18
16	24	37	68	34	66	60	38	77	85
87	78	59	46	57	52	47	42	23	14
12	22	40	56	43	50	53	61	79	89
11	80	39	49	54	55	44	62	21	90
93	81	65	51	48	45	58	36	20	8
6	19	63	33	67	35	41	64	82	95
97	75	69	70	30	29	28	27	76	4
91	1	99	3	96	7	13	86	17	92

Fig. 68

Remark. Our choice for the place of 1 being here the same as in the Arabic, right-to-left squares, the horizontal rows of evenly-odd orders are, except for their end elements, the same in the original version and in the transcription. (We have chosen to start, as for the other squares, next to the lower left-hand corner.)

Mathematical basis of this method

The aim when constructing borders of even orders, whether or not divisible by 4, is in principle the same: to reach a situation where only ‘neutral placings’ are needed, that is, taking tetrads of consecutive numbers of which the extremes are put in one row and the middle ones in the opposite row: indeed (Fig. 69), with the complements of these tetrads in place, not only will the sum due be obtained in cells facing one another, but also in all groups of four cells in each row. The same principle had been used for completing the borders in odd-order squares displaying separation by parity (above, p. 184, top).

For such a situation to be attained in the borders of even order, an initial placing is required which must satisfy the following conditions.

- (i) The initial placing must settle the question of the corner cells, each common to two rows.
- (ii) The initial placing must be such that, after addition of the complements, each row displays its sum due, thus $m \cdot \frac{n^2+1}{2}$ if m cells are filled.
- (iii) This number m of cells filled must be as low as possible in order for the method to be applicable, as far as possible, to the smallest orders.
- (iv) The initial placing must be uniform to be applicable to any order of the same type, excepting at most small orders.
- (v) The number of cells left empty thereafter must be divisible by 4 for subsequent neutral placings.

Note also that, ideally, the numbers placed preliminarily should be the first natural numbers, for that will make the neutral placings straightforward. But this condition may be disregarded if the preliminary placing made with other numbers is particularly easy to remember.

	α	$n^2 + 1 - (\alpha + 1)$	$n^2 + 1 - (\alpha + 2)$	$\alpha + 3$	
	$n^2 + 1 - \alpha$	$\alpha + 1$	$\alpha + 2$	$n^2 + 1 - (\alpha + 3)$	

Fig. 69

Consider, in accordance with the aim of being left with neutral placings, the situation for an order $n = 4k + 2$. The smallest possibility, also applicable to the smallest order, 6, is to put ten smaller numbers, three in each row (including two consecutive corners); with the complements in place, this will leave, if the order in question is $n = 4k + 2$ with $k > 1$, $4(k - 1)$ empty cells in each row.

Suppose then we place the ten smaller numbers as in Fig. 70. Let us designate by a prime their complements, that of a being thus $a' = (n^2 + 1) - a$. After writing in the complements, we must have the sum due for six cells filled, thus, for the upper and left-hand rows, respectively,

$$a + b + c + d' + e' + f' = 3(n^2 + 1),$$

$$a + g + h + c' + i' + j' = 3(n^2 + 1).$$

Since $d' + e' + f' = 3(n^2 + 1) - d - e - f$ and $c' + i' + j' = 3(n^2 + 1) - c - i - j$, the two previous relations become

$$\begin{cases} a + b + c = d + e + f \\ a + g + h = c + i + j. \end{cases}$$

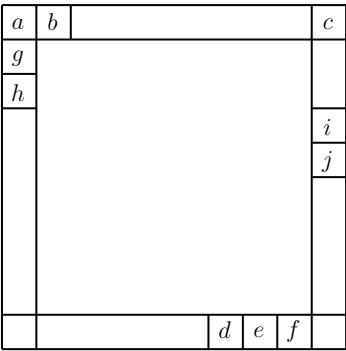


Fig. 70

This could be easily satisfied with the first ten numbers, for there are 140 possibilities.²¹⁶

13	196	2	194	5	190	9	186	180	19	176	23	172	14
171													26
24													173
175													22
20													177
179													18
16													181
15													182
185													12
10													187
189													8
6													191
193													4
183	1	195	3	192	7	188	11	17	178	21	174	25	184

Fig. 71

Now the method used in our text also relies on the preliminary placing of ten numbers, but, except for $n = 6$, discontinuously. Those of the upper row are 2, $n - 1$ and n , those of the lower row 1, $n - 3$ and $n + 3$, thus 2, 9, 10 and 1, 7, 13 in Fig. 68 or 2, 13, 14 & 1, 11, 17 in Fig. 71 ($n = 14$);

²¹⁶ Listed in our *Magic squares in the tenth century*, pp. 63–64. There are several examples of such possibilities in later Arabic treatises, see *Magic squares*, p. 172.

since their sums are equal, we shall have the sum due after adding the complements. The three numbers in the columns are $n - 1$, $n + 1$, $n + 2$ on the left and n , $n - 2$, $n + 4$ on the right, thus 9, 11, 12 and 10, 8, 14 in Fig. 68 or 13, 15, 16 & 14, 12, 18 in Fig. 71, both sets making equal sums. The cyclical placing distributes the remaining numbers appropriately: the upper row associates 5 and $2n - 5$, 9 and $2n - 9$, and so on to $n - 5$ and $2n - (n - 5)$, whereby the row is increased by $k - 1$ times $2n$; the lower row associates the other odd numbers, thus 3 and $2n - 3$, 7 and $2n - 7$, and so on to $n - 7$ and $2n - (n - 7)$, whereby here too the sum in the row is increased by $(k - 1)2n$. The columns will likewise comprise, together with the initial numbers, the pairs 6 and $2n - 4$, 10 and $2n - 8$, ..., $n - 4$ and $2n - (n - 6)$ for one of them, and for the other the remaining even numbers, thus 4 and $2n - 2$, 8 and $2n - 6$, ..., $n - 6$ and $2n - (n - 8)$; this adds to each column two sequences of $k - 1$ numbers making the same sum, namely $(k - 1)(2n + 2)$. Since the sums in these two opposite rows are equal, the complements of one will produce in the other the sum due.

Remark. If $n = 4k + 2$ with $k \geq 2$, we could just keep the configuration seen in the 6×6 square for the first ten numbers and complete the rows with neutral placings. But then the order number would not appear in an upper corner cell (see p. 205).

B. Bordered squares of evenly-even orders ($n = 4k$)

11	22	124	125	19	18	17	129	130	131	132	12
10	31	122	24	120	27	116	110	37	106	32	135
136	105	47	54	53	93	94	95	96	48	40	9
8	38	46	59	90	56	88	82	60	99	107	137
138	109	100	81	68	79	74	69	64	45	36	7
139	34	44	62	78	65	72	75	83	101	111	6
5	33	102	61	71	76	77	66	84	43	112	140
4	115	103	87	73	70	67	80	58	42	30	141
142	28	41	85	55	89	57	63	86	104	117	3
143	119	97	91	92	52	51	50	49	98	26	2
1	113	23	121	25	118	29	35	108	39	114	144
133	123	21	20	126	127	128	16	15	14	13	134

Fig. 72

7	14	13	53	54	55	56	8
6	19	50	16	48	42	20	59
60	41	28	39	34	29	24	5
4	22	38	25	32	35	43	61
62	21	31	36	37	26	44	3
63	47	33	30	27	40	18	2
1	45	15	49	17	23	46	64
57	51	52	12	11	10	9	58

Fig. 73

(§§ 40–43) Method of construction

For a general evenly-even square, our author takes the smaller numbers to be placed in the outer border in descending order, for he supposes the inner part to have been filled first (Fig. 72, here from left to right, with $n = 12$, and 22 the first number to be placed since the outer border comprises 44 cells). He takes groups of four consecutive numbers of which he places the extremes (say) on the top and the two middle on the bottom, until he is left with six empty cells within the row. Of the next six numbers, the first two are put above and the others below — leaving, as usual, the opposite cells empty. He then fills, successively, the two corners, the cell below the corner just filled, then one cell on the other side. Returning to the opposite column, he completes the two columns by means of groups of four consecutive numbers, the two extremes on one side and the two middle on the other. These smaller numbers and their complements fill the outer border completely.

The case of order 8 is similar, except that we directly begin by putting two numbers at the top and four at the bottom before filling the corner cells (Fig. 73). These are the two squares represented in our treatise.

Mathematical basis of this method

As before, we shall consider which preliminary placing could, in the case of evenly-even orders $n = 4k$, leave us with neutral placings, thus a quantity of empty cells equal to a multiple of 4. Consider then a square of order $n = 4k$ with $k \geq 2$.

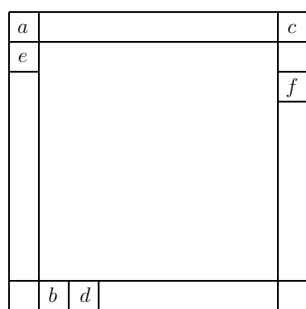


Fig. 74

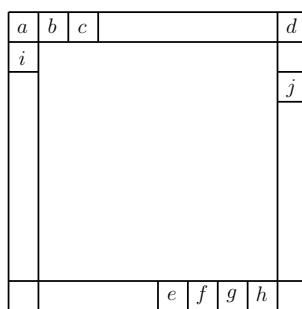


Fig. 75

We shall be left with a quantity of empty cells divisible by 4 by placing two smaller numbers in each row, two of which must occupy consecutive corner cells (Fig. 74). Indeed, since with the complements in place four cells in each row will be occupied, $4(k - 1)$ will still be empty. Thus only six smaller numbers are needed for this preliminary placing.

We might also place two more numbers in one row and two more in its opposite; this will leave $4(k-2)$ empty cells in these two rows and $4(k-1)$ in the other two, as before. Here we shall have placed ten smaller numbers (Fig. 75). This method will also be general since it is applicable to the smallest square, of order 8.

This remains true if we place another pair of smaller numbers in the columns, thus altogether fourteen numbers. We could of course continue placing further pairs, but this would be at the cost of simplicity and no longer be applicable to smaller orders.

Suppose we place six numbers as in Fig. 74. Let us once again designate by a prime their complements. After writing in the complements, we must have the sum due for four cells filled, thus, for the upper and left-hand rows, respectively,

$$a + b' + d' + c = 2(n^2 + 1),$$

$$a + c' + f' + e = 2(n^2 + 1).$$

Since $b' + d' = 2(n^2 + 1) - b - d$ and $c' + f' = 2(n^2 + 1) - c - f$, the two previous relations become

$$\begin{cases} a + c = b + d \\ a + e = c + f, \end{cases}$$

a and c being common to the two equalities since they occupy the corner cells. We shall have attained our aim if we can solve this pair of equations by means of the first six natural numbers. The remainder of the border, each row of which contains $4(k-1)$ empty cells, will be filled by means of tetrads of consecutive numbers. But the possibilities offered by this placing of the first six numbers are limited: we may either place 2 and 3 in the corners and 1 and 4 in the opposite row, or 4 and 5 in the corners and 3 and 6 in the facing row, the two remaining numbers being used in both cases to cancel out the difference of 1 left in the columns. At least this preliminary placing is particularly easy to remember.

The placing of ten numbers offers notably more possibilities. In that case we have the relations (Fig. 75)

$$\begin{cases} a + b + c + d = e + f + g + h \\ a + i = d + j, \end{cases}$$

which gives, with the first ten numbers, 128 solutions.²¹⁷

The method described in our text relies on the preliminary placing of ten numbers, but which are just not the first ones, namely, in Fig. 72,

²¹⁷ Listed in *Magic squares in the tenth century*, pp. 57–58.

$$11 + 18 + 17 + 12 = 16 + 15 + 14 + 13,$$

$$11 + 10 = 12 + 9.$$

Generally, we shall write in the top line $n - 1$, n , $n + 5$, $n + 6$ and at the bottom $n + 1$, $n + 2$, $n + 3$, $n + 4$, which make equal sums; in the columns, along with $n - 1$, we shall put $n - 2$ and, along with n , $n - 3$, which make equal sums as well. The remaining smaller numbers, from 1 to $n - 4$ and from $n + 7$ to $2n - 2$, are then just written as tetrads of consecutive numbers, with neutral placings.

As to the square of order 8 (Fig. 73), which differs in the description of the method (see above), it simply follows the same rule. By the way, its initial arrangement, with the same small numbers, could just be kept for any order $n = 4k$ and the rows completed with neutral placings of the subsequent numbers. The remark made above for the evenly-odd orders (p. 209) also applies, *mutatis mutandis*, here.

<i>xi</i>	<i>xxii</i>	.	.	<i>xi</i>	<i>xviii</i>	<i>xvii</i>	.	.	.	.	<i>xii</i>
<i>x</i>	<i>ix</i>	.	<i>ii</i>	.	<i>v</i>	.	.	<i>xv</i>	.	<i>x</i>	.
.	.	<i>vii</i>	<i>xiv</i>	<i>xiii</i>	.	.	.	.	<i>viii</i>	<i>xviii</i>	<i>ix</i>
<i>viii</i>	<i>xvi</i>	<i>vi</i>	<i>v</i>	.	<i>ii</i>	.	.	<i>vi</i>	.	.	.
.	.	.	.	<i>iv</i>	.	.	<i>v</i>	<i>x</i>	<i>v</i>	<i>xiv</i>	<i>vii</i>
.	<i>xii</i>	<i>iv</i>	<i>viii</i>	.	<i>i</i>	<i>viii</i>	.	.	.	.	<i>vi</i>
<i>v</i>	<i>xi</i>	.	<i>vii</i>	<i>vii</i>	.	.	<i>ii</i>	.	<i>iii</i>	.	.
<i>iv</i>	.	.	.	.	<i>vi</i>	<i>iii</i>	.	<i>iv</i>	<i>ii</i>	<i>viii</i>	.
.	<i>vi</i>	<i>i</i>	.	<i>i</i>	.	<i>iii</i>	<i>ix</i>	.	.	.	<i>iii</i>
.	.	.	.	.	<i>xii</i>	<i>xi</i>	<i>x</i>	<i>ix</i>	.	<i>iv</i>	<i>ii</i>
<i>i</i>	.	<i>i</i>	.	<i>iii</i>	.	<i>vii</i>	<i>xiii</i>	.	<i>xvii</i>	.	.
.	.	<i>xxi</i>	<i>xx</i>	.	.	.	<i>xvi</i>	<i>xv</i>	<i>xiv</i>	<i>xiii</i>	.

Fig. 76

This part on common bordered magic squares of even orders is, as Part I on common bordered magic squares of odd orders, omitted by MS. $\mathcal{D}$ and thus preserved only in Anṭākī's text. The reason is the same in both cases, namely the existence of a synthetic representation (*Khāzinī*'s) displaying the arrangement in the first borders of even orders (Fig. 76²¹⁸). Notable is the place of the smallest number, each time next to the lower left-hand corner (for us), while the order numbers occupy the opposite corners; the 4×4 square is the sole exception.

²¹⁸ Here adapted: see, for the original, MS. $\mathcal{D}$, fol. 63^r.

Part IV: Construction of even-order composite squares

Introductory note on composite squares

Here the construction of a larger magic square is reduced to that of its individual parts, mostly squares, each of which displays its sum due. The possibility of constructing a square in such a way depends of course on the divisibility of its order. A prime order is thus excluded.

(a) What is usually meant by composite square is a square divided into even or odd-order subsquares each of the same size, *filled one after the other completely by a continuous sequence of numbers in magic arrangement*. Since the sums in the subsquares are different, but they form an arithmetical progression, these subsquares must be arranged in such a way that the main square will itself be magic. Now, because of the impossibility of constructing a magic square of order 2 with different elements, the order of such a composite square cannot be twice a prime nor twice the smallest possible magic squares. This last restriction excludes both orders $n = 6$ and $n = 8$, and the smallest such composite square is thus of order 9, broken up into nine squares of order 3 (Fig. 77 & Fig. 78). The next possible order is $n = 12$, which may be divided into sixteen squares of order 3 arranged according to one of the ways for order 4, or into nine squares of order 4 placed according to the arrangement for order 3. These two examples are given in the tenth century by Abū'l-Wafā' Būzjānī.²¹⁹

2	9	4
7	5	3
6	1	8

Fig. 77

11	18	13	74	81	76	29	36	31
16	14	12	79	77	75	34	32	30
15	10	17	78	73	80	33	28	35
56	63	58	38	45	40	20	27	22
61	59	57	43	41	39	25	23	21
60	55	62	42	37	44	24	19	26
47	54	49	2	9	4	65	72	67
52	50	48	7	5	3	70	68	66
51	46	53	6	1	8	69	64	71

Fig. 78

There is no mention of such squares in our treatise. On the other hand, the two subsequent types are fully explained in it.

²¹⁹ See the edition of his text, pp. 183–185 and (Arabic) 238–240.

(b) If the main square can be divided into subsquares of the same *even* order ≥ 4 , these subsquares can be *filled individually with pairs of complements in magic arrangement*. Since they are all of the same size and display the same sum, they can be arranged in any manner. Note that in this case, an 8×8 square is possible, as observed, or reported, by the author of MS. *D* (see above, p. 125). Note also that such squares are generally of order $n = 4k$, but may be of evenly-odd order as long as that order has a factor (smaller than $\frac{n}{2}$) other than 2.

(c) A less conventional possibility is to divide the main square into *unequal parts*, square or even rectangular, but here again filled with pairs of complements. Each part is to display its sum due, which of course will vary according to its dimensions. This extends the possibility of constructing composite squares to evenly-odd orders not satisfying the above condition, beginning in our treatise with $n = 10$.

A. Division into equal subsquares

(§§ 44–45, 48) Consider thus the division of a given square into identical subsquares of even orders which all display their sum due *with respect to the whole square*, thus $m \cdot \frac{n^2+1}{2}$ for $m \times m$ subsquares within a square of order n . In the examples given, the order of the main square is either of the form $n = 4k$, with subsquares of orders 4, 6, 8 (§ 44), or n is evenly-odd, but of the form $2 \cdot r$ with r a composite (odd) number, say $r = s \cdot t$ with $s, t \geq 3$, for the square may then be divided into s^2 squares of even order $2t$ (§ 45). Thus we find:

— A 12×12 square divided into four 6×6 (bordered) subsquares (see example p. 127).

— A 12×12 square divided into nine 4×4 subsquares (p. 129).

— A 16×16 square divided into four 8×8 (bordered) subsquares (p. 131).

— A 16×16 square divided into sixteen 4×4 subsquares (p. 133).

— A 18×18 square divided into nine 6×6 (bordered) subsquares (p. 135). In addition to considering (and constructing) the case $n = 18$, our text mentions the division of a 30×30 square (this being the next admissible evenly-odd order) into nine 10×10 subsquares or twenty-five 6×6 subsquares (§ 45).

— A 20×20 square divided into twenty-five 4×4 subsquares (p. 137).

For filling the main square, as asserted in § 48, we shall take any subsquare and put in it the first set of smaller numbers, starting with the

two medians of the main square and their neighbours, then proceed with any other subsquare, and so on until the main square is filled: since each subsquare makes the same sum, their arrangement is indifferent. The smallest possibility is the 8×8 square divided into four 4×4 subsquares, omitted in our text but supplied in MS. $\mathcal{D}$ (p. 125).

B. Division into unequal parts

(§§ 44, 46–47, 49–54) A division into equal subsquares of even orders is not always necessary and not always possible. The impossibility includes first all the squares of odd orders $n = 2k + 1$ (§47) and second those squares of evenly-odd orders divisible by 2 only, as seen previously; in this latter case a division is still possible, but either into unequal parts (square or rectangular) or with a cross in the middle. Division into unequal parts is also possible for orders $n = 4k$, depending on the divisibility of n .

(§49) Consider the less banal case, that of evenly-odd orders. Suppose thus that the square to be constructed has the order $n = 4k + 2$. Draw in its centre a square of order $4t + 2$ ($k > t \geq 1$). The whole square is then broken up as follows (Fig. 79): in its centre, the square of order $4t + 2$, distant by $2(k - t)$ from the sides; in each corner, one square the side of which, $2(k - t)$, is divisible by 4 [or becomes so when taken together with that of the square opposite]; finally, four $2(k - t) \times (4t + 2)$ lateral rectangles, *not containing any part of the main diagonals* and with one of their dimensions [or that of two opposite rectangles] divisible by 4.²²⁰

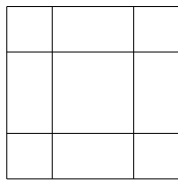


Fig. 79

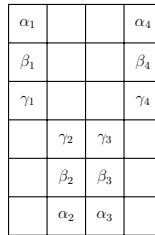


Fig. 80

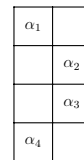


Fig. 81

We shall fill the square parts, central or in the corners, as before, with pairs of complements. We are then left with the rectangles. We may either fill them with neutral placings only, since their length, $4t + 2$, is even while their width, $2(k - t)$ [or twice it], is divisible by 4 (Fig. 80); or we can consider subsquares (which can always be 4×4) and fill them in such a way that their sum due appears, which will leave us again with

²²⁰ The cases between square brackets are not considered in our text. Thus its smallest example is $n = 14$ and not $n = 10$.

filling rectangular strips using neutral placings (Fig. 81).²²¹ Our text mentions (§ 46, p. 109) two examples of division, and constructs these and others, with subsquares of orders 4, 6, 8, 10 (interpolated mention of the case $n = 30$ in § 45). Thus we find:

— A 14×14 square with a central 6×6 square surrounded by eight 4×4 squares and four pairs of four-cells-long strips (§ 46 & p. 139).

— A 16×16 square with four 6×6 squares in the corners, a central 4×4 square and four 4×4 squares separated from the central square by four 2×4 rectangles (p. 141).

— A 16×16 square with a central 8×8 square surrounded by twelve 4×4 squares (p. 143).

— A 18×18 square with a central 10×10 square surrounded by twelve 4×4 squares and four pairs of four-cells-long strips (§ 46 & p. 145).

— A 20×20 square with four 8×8 squares in the corners separated by nine 4×4 squares, one of which in the centre (p. 147).

— A 20×20 square with four 6×6 squares in the centre and sixteen 4×4 squares around (p. 149; note the reduction of the central square: we thus remain with even orders ≤ 10 , see *n.* 156, p. 107).

— A 20×20 square with four 4×4 squares in the centre and, separated by four pairs of six-cells-long strips, eight 6×6 squares (p. 151).

— A 20×20 square with a central 8×8 square, four 6×6 squares in the corners, and eight 4×4 squares each completed with a 2×4 rectangle (p. 153).

C. Cross in the middle

(§ 50–54) Another possibility for evenly-odd orders is to eliminate a pair of median horizontal and vertical rows—in other words, to fill them separately (Fig. 82). Filling this cross (and taking into account that its centre meets the diagonals) will leave us with four squares of even order $2k$, the filling of which is well known.

Since the order of the whole square is $4k + 2$, it will contain the numbers from 1 to $n^2 = 16k^2 + 16k + 4$. We shall use $16k + 4$ of these numbers for the cross, for example those in the middle of the sequence, while the $8k^2$ first and the $8k^2$ last will fill the subsquares. How to fill the cross is explained in detail in the text (§§ 50, 52). For equalizing the cross, the branches of which are two cells wide, we shall put in each pair

²²¹ There are two examples of the second case, none of the first (though the arrangement of Fig. 80 is treated in § 49).

of such cells a pair of complements. This will give the sum due for two cells and we shall no longer need to consider the cross when filling the subsquares.

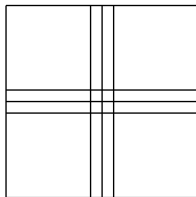


Fig. 82

We are still to obtain the sum due longitudinally in the branches and diametrically in the four central cells. Now it is already possible to attain the sum due with a length of six cells, comprising the cells in the middle of the cross, thus altogether twenty cells, by using ten consecutive pairs among the $8k + 2$ pairs to be placed in the cross. Consider first these $16k + 4$ numbers, aligned vertically by pairs of complements adding up to $n^2 + 1 = 16k^2 + 16k + 5$:

$$\begin{array}{cccccc} 8k^2 + 1 & 8k^2 + 2 & \dots & 8k^2 + 8k + 1 & 8k^2 + 8k + 2 \\ 8k^2 + 16k + 4 & 8k^2 + 16k + 3 & \dots & 8k^2 + 8k + 4 & 8k^2 + 8k + 3. \end{array}$$

Let us for instance take the last ten pairs, which may also be expressed as

$$\begin{array}{cccccccccc} \frac{n^2}{2} - 9 & \frac{n^2}{2} - 8 & \frac{n^2}{2} - 7 & \frac{n^2}{2} - 6 & \frac{n^2}{2} - 5 & \frac{n^2}{2} - 4 & \frac{n^2}{2} - 3 & \frac{n^2}{2} - 2 & \frac{n^2}{2} - 1 & \frac{n^2}{2} \\ \frac{n^2}{2} + 10 & \frac{n^2}{2} + 9 & \frac{n^2}{2} + 8 & \frac{n^2}{2} + 7 & \frac{n^2}{2} + 6 & \frac{n^2}{2} + 5 & \frac{n^2}{2} + 4 & \frac{n^2}{2} + 3 & \frac{n^2}{2} + 2 & \frac{n^2}{2} + 1 \\ \text{X} & \text{IX} & \text{VIII} & \text{VII} & \text{VI} & \text{V} & \text{IV} & \text{III} & \text{II} & \text{I} \end{array}$$

We shall place I and III in the centre, with the elements of each pair diagonally opposite. Next, the elements of the pairs II, IV, V, VI will occupy the vertical branch and the others the horizontal branch, as represented in Fig. 83.

With this arrangement, the occupied cells display in all three directions the sum due, namely:

- diagonally: $n^2 + 1$ for two cells,
- vertically: $3(n^2 + 1)$ for six cells,
- horizontally: $3(n^2 + 1)$ for six cells.

We are left with placing in the branches of the cross the $16k - 16$ remaining numbers, thus two sequences of $8(k - 1)$ complementary numbers, each sequence being formed by consecutive numbers since the centre

of the cross has been filled with the last ten pairs. Filling the remainder of the cross can thus be completed using neutral placings. It is also mentioned in our text (§§ 53, 54) that the preliminary equalization of the cross may be performed for orders $n = 4k + 2$ with k even on just one side of the branch, the effect being that we shall have an integral number of neutral placings on each side.²²²

Filling the part outside the cross is easy: we shall fill the four squares with the remaining numbers, disregarding those already used for the cross.

Remark. Thus, as usual, our text takes the numbers in decreasing order and starts with filling the cross.²²³

			$\frac{n^2}{2}+4$	$\frac{n^2}{2}-3$			
			$\frac{n^2}{2}-1$	$\frac{n^2}{2}+2$			
$\frac{n^2}{2}-7$	$\frac{n^2}{2}+7$	$\frac{n^2}{2}$	$\frac{n^2}{2}+3$	$\frac{n^2}{2}+9$	$\frac{n^2}{2}-9$		
$\frac{n^2}{2}+8$	$\frac{n^2}{2}-6$	$\frac{n^2}{2}-2$	$\frac{n^2}{2}+1$	$\frac{n^2}{2}-8$	$\frac{n^2}{2}+10$		
		$\frac{n^2}{2}-4$	$\frac{n^2}{2}+5$				
		$\frac{n^2}{2}+6$	$\frac{n^2}{2}-5$				

Fig. 83

Our text includes several examples of squares with a central cross.²²⁴

— A 10×10 square with a cross and four 4×4 subsquares (§ 46 & p. 155).

— A 14×14 square with a cross and four 6×6 subsquares (§ 46 & p. 157, with the remaining rectangular strips in the cross filled individually).

²²² This is not necessary. Furthermore, there is one example of neutral placings separated by a square (p. 151).

²²³ Some later authors, instead of filling the cross with the numbers in the middle, do it with the first and the last ones; see *Magic squares*, pp. 101–102 (*Les carrés magiques*, p. 98; *Маг. квадраты*, pp. 109–110). In that case, the placing is similar to that for the outer border of evenly-odd squares (the four numbers in the centre of the cross are then the four numbers in the corner cells).

²²⁴ The case $n = 6$ can also be treated in this way, the remaining 2×2 parts being then filled as a 4×4 square. See *Magic squares*, p. 101 (*Les carrés magiques*, p. 97; *Маг. квадраты*, p. 108), or *Magic squares in the tenth century*, p. 112.

— A 18×18 square with a cross and four 8×8 subsquares in the corners (§46 & p. 159).

Of course, we may also consider in this latter case a further division:

— A 18×18 square with a cross and sixteen 4×4 subsquares (p. 161).

Remark. Knowing how to fill the cross as well as 4×4 squares displaying equal sums is sufficient to construct *any even-order square*. Indeed, if $n = 4k$, the square may be filled with squares of order 4, and if $n = 4k + 2$ the cross will leave strips of k squares of order 4, one of which will be cut by the cross when k is odd. But there are other possibilities in this last case of evenly-odd squares, according to the form of k : if $k = 3t$, thus $n = 12t + 2$, we may also consider strips of t squares of order 6 (displaying equal sums) on both sides of the cross (example of p. 157, $n = 14$); if $k = 3t + 1$, thus $n = 12t + 6$, we may also divide the square into strips of order 6 without using the cross (example of p. 135, $n = 18$, and mention of the case $n = 30$ in §45). This may explain the assertion of an eleventh-century author who, after explaining the method of the cross, writes that knowing how to fill the cross and both the 4×4 and the 6×6 squares enables us to construct any square of even order.²²⁵

²²⁵ See *Un traité médiéval*, pp. 83 & 165. This (unknown) author's knowledge of our text seems apparent from two other places: his repeating an argument in §47 (above, p. 111, *n.* 166) and his possible allusion to Part II (above, p. 174).

V. Arabic glossary

ارثماطيقى : *(line)* 2.

اسحاق : 5.

اقلیدس : 4.

الحجازی : 2.

المالهانی : 8.

النوبختی : 8.

نیقوماخس : 2.

ا

ابدأ : 18, 66, 69, 74, 81, 83, 95,
201, 220, 255,

أتی (v.) : (I) 6, 92, 116, 118, 120,
122, 129, 145, 189, 392, 445,
476, 479, 510, 580, 585.

مأتی : 245.

أخذ (v.) : 69, 71, 74, 80, 83, 84,
110, 124, [271], 321, 363, 448,
451, 480, 575, 578, 599.

آخر : [488].

آخر : 15, 21, 99, 106, 106, 166,
192, 252, 254, 281,

اخیر : 81, 145, 147, 247, 250,
[275], [276], [277], 295, 383,
449, 509.

أدى (v.) : (II) 95, 526, 533, 544,
564.

ارضة : 7.

ازاء (بازاء) : 24, 26, 28, 30, 35, 38,
40, 43, 51, 52,

الآ : 456, 457, [569].

ألف (v.) : (II) 247.

تألیف : 244, 244, 299.

أما : 22, 32, 110, 197, 205, 231,
301, . . . ; (إما) [262-3], [398-9],
485, 588.

أمر (s.) : 1, 19, 230.

انت : [261], 473.

اتما : [261], 375, 588.

اول : 8, 141, 143, 150, 164, 172,
186, 188, 206, . . . ; (s.) 1,
[378]; (اولاً) 580.

ای : 280.

ایضاً : 279, 416, 430, 462, 505,
536, 537, [538], 542, 543,

ب

بدأ (v.) : (I) 469. (VIII) 3, 16.

مبتدیء : 20.

مبتدأ : 456, 457, 472.

بطل (v.) : (I) [573].

بعد (v.) : (III) 285, [291].

بعد : [89], [132], 202, [228], 235,
259, 321, 324, [390],

بعض : 10, 18, 100, 100, 374, 374,
518, 518, 519, [573], [573].

بقي (v.): (I) 45, 64, 89, 197, 199,
228, 231, [265], [266], 302,
359, 363,

باقٍ : 59, 65, 65, 94, 116, 378,
393, 447, 586, 600, 626.

بلغ (v.): (I) 17, 469.

باب : 20, [196], [378], [585].

بيت : 28, 32, 34, 40, 41, 42, 45,
47, 48, 48,

بين (v.): (II) 229, [572], [584],
587.

بين : 286, [291], 560, 587.

ت

تَحْتِ : 26, 49, 50, 53, 54, 56, 58,
70,

ترك (v.): (I) 598, 609.

متعِب : 9.

تلف (v.): (I) 10.

تلو (v.): (I) 63, 71, 80, 85, 121,
127, 136, See p. 21n.

تالٍ : 33.

تمّ (v.): (I) 31, 44, 60, 441.

تمام : 459, 605.

تامّ : 575, 582.

ث

ثبت (v.): (IV) 13, 39, 41, 47,
49, 51, 52, 53, 54,

اثبات : 110, 190.

مثَلث : 105.

ثُمَّ : 2, 4, 5, 6, 18, 19, 25, 27, 29,
33, 36,

مَثْنَى : [569].

ج

جَدًّا : 9.

جدول : 1, 2, 4, 5.

جزء : 530, 530, 531, 531, 544.

جعل (v.): (I) 23, 25, 26, 27, 29,
32, 33, 35, 35, 36,

جمع (v.): [272], [272], [273],
[274], [275], [276], [277], 332,
335, 339, 341,

جميع : 19, 79, 92, 95, 109, 122,
129, 241, [300], 327, 372,

مجموع : [273], [278].

جمل (v.): (IV) 224, 521, 522,
547, 548, 550, 551, 554, 554.

جملة : 13, [14], 107, [222], 521,
521, 522, 535, ...; جملةً [526].

جانب : 50, 51, 54, 59, 72, 75,
105, 105, 109,

جوز (v.): (I) 196. (III) [532].
(VI) 312.

ح

حتّى : 7, 11, 79, 92, 103, 112,
116, 118, 120, 122, 131,

حَسَبَ (على) : 227.

حشو (v.): (I) 96, 126, 130, 314, 384.

حشو (s.): 92, 94, 95, 493.

حصل (v.): (I) 470, 484.

حقّ : 207, 209, 222, [224], 226, 237, 237, 239, 240, 252,

حوج (v.): (VIII) 191, 227, 244, 245, [261], [266], 279, 299, 565, 566, 566, 604, 624.

حوط (v.): (IV) 64, 553, 556.

محيط : 198, 232, 384.

حيث : 17, 34, 37, 63, 322, 324, 444, 445, 577.

حال : 357.

حيلة : [532].

حينئذٍ : 108, 191, 363, 600, 619.

خ

خرج (v.) : (I) [399]. (X) 10.

استخراج : 9.

خارج : 446.

خزانة : 6.

خاصّةً : [529].

مختصر : 7.

خط : 8, 8.

خلاف : 192.

مختلف : 517, 544.

د

داخل : 40, 108, 123, 126, 128, 130, 198, 232, 314,

تدرّج : 190.

دلّ (v.): (X) 421, 561, 632.

دائمًا : 95, 112, 122.

دون : 6, 47.

ذ

ذكر (v.): (I) 1, [300], 533.

ر

مرّج : 12 (adj.), 13, 23, 32, 35, 40, 64, 93, 94, 95, 96, 97,

مرّبة (s.): 31, 44, 45, 45, 61, 62, 63, 65, 111, 113, 122, 442.

رتب (v.): (II) 103, 520, 533, 541.

ترتيب : 97, 191, 207, 227, 228, [261], [262], 295, 632.

رسم (v.): (I) 12, 18, 94, 99, 101, 106, 121, 147, 280, 281, 281, 282, 283, 284, (VIII) 92.

رسم (s.): 99, 101, 577.

راد (v.): (IV) 17, 145, 189, 190, 476, 562, 574, 580.

ز

زوج : 66, 68, 72, 78, 81, 85, 89, 100, 106, ... ; (pl. زوجيّة) 109, 192, 227.

زوج زوج : 422, 477, 523.

زوج فرد : 423, 443, [479], 525, [528], 529, 543.

زوج زوج فرد : 423, 477, 523-4.

زاوية : 28, 28, 40, 42, 48, 49, 50, 52, 52, 55,

زيد (v.): (I) 193, 194, 201, 207,
208, 209, 210, 211,

زيادة : 18, 67, 97, 222, 227, 229,
242, 246, ... ; زيادةً 356, 362,
396, 402.

س

مسرى : [427].

سطح (s.): 12.

سطر : 14, 15, 18, 19, 19, 575,
581, 584, 600, 601,

اسفل : 49, 56, 58, 68, 74, 79, 88,
90, ... ; (من اسفل) 142, 144,
492, 494; (اسفل) 157.

سقط (v.): (IV) [570], [571].

اسقاط : [573].

سلك (v.): (I) 94.

سلم (v.): (II) 10.

اسم : 458, 463, [463], 500.

اسهل : 21.

مساو : 105, 403.

متساو : 107, 512.

سائر : 205, 216, 243, 419, 559.

ش

شبه (v.): (IV) 383, 530.

شبيه : 104, 105.

شرح (v.): (I) 302, 378, 396.

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